MATHEMATICS

Mob. : 9470844028
9546359990

AIM POINTS
MATHEMATICS
DIR. FIROZ AHMAD

Ramrajya More, Siwan (Bihar)

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CONTINUITY

& Their Properties

CONTENTS

THINGS TO REMEMBER

1. A real valued function f(x) is continuous at a point 'a' in its domain iff

$\lim_{x \to a^-} f(x) = f(a)$ $\lim_{x \to a^+} f(x) = f(a)$

i.e. the limit of the function at x = a is equal to the value of the function at x = a.

2. A functionf(x) is said to be continuous if it is continuous at every point of its domain.
3. Sum, difference, product and quotient of continuous functions are continuous i.e. if f(x) and g(x) are continuous functions on their common domain, then $f \pm g$, fg, $\frac{f}{g}$, kf (k is a constant) are continuous.
4. Let f and g be real functions such that fog is defined. If g is continuous at x = a and f is continuous at g(a), then fog is continuous at x = a.
5. Following functions are everywhere continuous :
   (i) A constant function
   (ii) The identity function
   (iii) A polynomial function
   (iv) Modulus function
   (v) Exponential function
   (vi) Sine and Cosine functions
6. Following functions are continuous in their domains :
   (i) A logarithmic function
   (ii) A rational function
   (iii) Tangent, cotangent, secant and cosecant functions.
7. If is continuous function, then | f | and $\frac{1}{f}$ are continuous in their domains.
8. $\sin^{-1}x$, $\cos^{-1}x$, $\tan^{-1}x$, $\cot^{-1}x$, $\text{cosec}^{-1}x$ and $\sec^{-1}x$ are continuous functions on their respective domains.

EXERCISE-1

1. Let f and g be two real functions, consinuous at x = a. Let α be a real number. Then f – g is continuous at x = a.
2. Examine the function f(t) given by

$$f(t) = \begin{cases} \dfrac{\cos t}{\dfrac{\pi}{2} - t} & ; t \neq \dfrac{\pi}{2} \\ 1 & ; t = \dfrac{\pi}{2} \end{cases}$$

for continuity at $t = \frac{\pi}{2}$

3. Show that the function f(x) = 2x – |x| is continuous at x = 0.
4. Discuss the continuity of the function of given by :

$f(x) = |x - 1| + |x - 2|$ at x = 1 and x = 2.

5. Determine the value of k for which the following function is continuous at x = 3.

$$f(x)=\begin{cases}\dfrac{x^2-9}{x-3}, & x\neq 3\\ k, & x=3\end{cases}$$

6. Find the value of the constant λ so that the function given below is continuous at x = –1.

$$f(x)=\begin{cases}\dfrac{x^2-2x-3}{x+1}, & x\neq 1\\ \lambda & ,x=-1\end{cases}$$

7. If the function f(x) defined by

$$f(x)=\begin{cases}\log(1+ax)-\log(1-bx) & ,\text{ if } x\neq 0\\ k & ,\text{ if } x=0\end{cases}$$

8. Find the values of a so that the function f(x) defined by

$$f(x)=\begin{cases}\dfrac{\sin^2 ax}{x^2}, & x\neq 0\\ 1 & ,x=0\end{cases}$$

may be continuous at x = 0.

9. If the function f(x) given by

$$f(x)=\begin{cases}3ax+b & ,\text{if } >1\\ 11 & ,\text{if } x=1\\ 5ax-2b & ,\text{if } x<1\end{cases}$$

is continuous at x = 1, find the values of a and b.

10. Prove that the greatest integer function [x] is continuous at all points except at integer points.

11. Le $f(x)=\begin{cases}\dfrac{1-\cos 4x}{x^2} & ,\text{if } x<0\\ a & ,\text{if } x=0\\ \dfrac{\sqrt{x}}{\sqrt{16+\sqrt{x}}-4} & ,\text{if } x>0\end{cases}$

Determine the value of a so that f(x) is continuous at x = 0.

12. Determine f(0) so that the function f(x) defined by

$$f(x) = \frac{(4^x - 1)^3}{\sin\frac{x}{4}\log\left(1+\frac{x^2}{3}\right)}$$

becomes continuous at x = 0.

13. Dicsuss the continuity of the following functions at the point(s) :

(i) $f(x) = \begin{cases} |x|\cos\left(\frac{1}{x}\right), x \neq 0 \\ 0 \quad , x = 0 \end{cases}$ at $x = 0$

(ii) $f(x) = \begin{cases} (x-a)\left(\frac{1}{x-a}\right), x \neq 0 \\ 0 \quad , x = 0 \end{cases}$ at $x = 0$

(iii) $f(x) = \begin{cases} \frac{1-x^n}{1-x}, x \neq 1 \\ n-1 \ , x = 1 \end{cases}$ $n \in N$ at $x = 1$

(iv) $f(x) = \begin{cases} \frac{2|x|+x^2}{x}, x \neq 0 \\ 0 \quad , x = 1 \end{cases}$ at $x = 0$

14. Show that $f(x) = \begin{cases} 1+x^2, \text{if } 0 \le x \le 1 \\ 2-x \ , \text{if } x > 1 \end{cases}$ is discontinuous at x = 1.

15. Examine the continuity of the function

$$f(x) = \begin{cases} 3x-2, x \le 0 \\ x+1 \ , x > 0 \end{cases} \text{ at } x = 0$$

also sketch the graph of this function.

16. For what value of k is the following function continuous at x = 1 ?

17. Determine the value of the constant k so that the function

$$f(x) = \begin{cases} kx^2, x \le 2 \\ 3 \ , x > 2 \end{cases} \text{ is discontinuous at } x = 1.$$

18. Prove that the function

$$f(x) = \begin{cases} \dfrac{x}{|x|+2x^2}, & x \neq 0 \\ k, & x > 0 \end{cases}$$

remain discontinuous at x = 0, regardless the choice of k.

19. Find the value of k if f(x) is continuous at $x = \frac{\pi}{2}$, where

$$f(x) = \begin{cases} \dfrac{k\cos x}{\pi - 2x}, & x \neq \dfrac{\pi}{2} \\ 3, & x = \dfrac{\pi}{2} \end{cases}$$

20. If $f(x) = \begin{cases} \dfrac{x-4}{|x-4|} + a, & \text{if } x < 4 \\ a+b, & \text{if } x = 4 \\ \dfrac{x-4}{|x-4|} + b, & \text{if } x > 4 \end{cases}$ is continuous at x = 4, find a, b.

21. For what value of k is the function

$$f(x) = \begin{cases} \dfrac{\sin 2x}{x}, & x \neq 0 \\ k, & x = 0 \end{cases}$$

continuous at x = 0 ?

22. If $f(x) = \begin{cases} \dfrac{2^{x+2} - 16}{4^x - 16}, & \text{if } x \neq 2 \\ k, & \text{if } x = 2 \end{cases}$ is continuous at x = 2, find k.

23. Extend the definition of the following by continuity

$f(x) = \dfrac{1-\cos 7(x-\pi)}{5(x-\pi)^2}$ at the point $x = \pi$.

24. In each of the following, find the value of the constant k so that the given function is continuous at the indicated point;

(i) $f(x) = \begin{cases} \dfrac{1-\cos 2kx}{x^2}, & \text{if } x \neq 0 \\ 8, & \text{if } x = 0 \end{cases}$ at $x = 0$

(ii) $f(x)=\begin{cases}(x-1)\tan\frac{\pi x}{2}, & \text{if } x\neq 1\\ k, & \text{if } x=1\end{cases}$ at $x=1$

(iii) $f(x)=\begin{cases}k(x^2-2x), & \text{if } x<0\\ \cos x, & \text{if } x\geq 1\end{cases}$ at $x=0$

(iv) $f(x)=\begin{cases}kx+1, & \text{if } x\leq \pi\\ \cos x, & \text{if } x>\pi\end{cases}$ at $x=\pi$

(v) $f(x)=\begin{cases}\frac{x^2-25}{x-5}, & \text{if } x\neq 5\\ k, & \text{if } x=5\end{cases}$ at $x=5$

(vi) $f(x)=\begin{cases}kx^2, & x\geq 5\\ 4, & x<1\end{cases}$ at $x=1$

(vii) $f(x)=\begin{cases}k(x^2+2), & \text{if } x\leq 0\\ 3x+1, & \text{if } x>0\end{cases}$

25. Discuss the continuity of the f(x) at the indicated points :
   (i) f(x) = |X| + |X – 1|, at x = 0, 1
   (ii) f(x) |x – 1| + |x + 1|, at x = –1, 1.

26. If f(x) $\begin{cases}2x^2+k, & \text{if } x\geq 0\\ -2x^2+k, & \text{if } x<0\end{cases}$, then what should be the value of k so that f(x) is continuous at x = 0.

27. For what value of k is the following function continuous at x = 2 ?

$$f(x)=\begin{cases}2x+1; & \text{if } x<2\\ k; & \text{if } x=2\\ 3x-1; & \text{if } x>2\end{cases}$$

28. Let $f(x)=\begin{cases}\frac{1-\sin^3 x}{3\cos^2 x}; & \text{if } x<\frac{\pi}{2}\\ a; & \text{if } x=\frac{\pi}{2}\\ \frac{b(1-\sin x)}{(\pi-2x)^2}; & \text{if } x>\frac{\pi}{2}\end{cases}$. If f(x) is continuous at x = $\frac{\pi}{2}$, find a and b.

29. If the functions f(x), defined below is continuous at x = 0, find the value of k :

$$f(x)=\begin{cases}\dfrac{1-\cos 2x}{2x^2} & ;x<0\\ k & ;x=0\\ \dfrac{x}{|x|} & ;x>0\end{cases}$$

30. The composition of two continuous functions is a continuous function.
31. A constant function is everywhere continuous.
32. The identity function is everywhere continuous.
33. The logarithmic function is constinuous in its domain.
34. The sine function is everywhere continuous.
35. The cosine function is everywhere continuous.
36. Discuss the continuity of the function f(x) given by

$$f(x)=\begin{cases}2x-1, \text{if } x<0\\ 2x+1, \text{if } x\geq 0\end{cases}$$

37. Determine the value of the constant k so that the function

$$f(x)=\begin{cases}kx^2, \text{if } x\leq 2\\ 3, \text{if } x<2\end{cases} \text{ is continuous.}$$

38. Find the points of discontinuity, if any, of the following functions :

(i) $f(x)=\begin{cases}\dfrac{x^4-16}{x-2}, \text{if } x\neq 2\\ 16, \text{if } x=2\end{cases}$

(ii) $f(x)=\begin{cases}\dfrac{\sin x}{x}, \text{if } x\neq 0\\ 2x+3, \text{if } x\geq 0\end{cases}$

(iii) $f(x)=\begin{cases}\dfrac{\sin x}{x}+\cos x, \text{if } x\neq 0\\ 5, \text{if } x\geq 0\end{cases}$

(iv) $f(x)=\begin{cases}\dfrac{e^x-1}{\log_e(1+2x)}, \text{if } x\neq 0\\ 7, \text{if } x=0\end{cases}$

(v) $f(x)=\begin{cases} |x-3| & ,\text{if } x\geq 1 \\ \dfrac{x^2}{4}-\dfrac{3x}{2}+\dfrac{13}{4} & ,\text{if } x<1 \end{cases}$

(vi) $f(x)=\begin{cases} 2x & ,\text{if } x<0 \\ 0 & ,\text{if } 0\leq x\leq 1 \\ 4x & ,\text{if } x>1 \end{cases}$

(vii) $f(x)=\begin{cases} -2 & ,\text{if } x\leq 1 \\ 2x & ,\text{if } -1<x<1 \\ 2 & ,\text{if } x\geq 1 \end{cases}$

39. The following, determine the value of constant involved in the definition so that the given function is continuous :

(i) $f(x)=\begin{cases} \dfrac{\sin 2x}{5x} & ,\text{if } x\neq 0 \\ 3k & ,\text{if } x=0 \end{cases}$

(ii) $f(x)=\begin{cases} \dfrac{k\cos x}{\pi-2x} & ,x<\dfrac{\pi}{2} \\ 3 & ,x=\dfrac{\pi}{2} \\ \dfrac{3\tan 2x}{2x-\pi} & ,x>\dfrac{\pi}{2} \end{cases}$

(iii) $f(x)=\begin{cases} kx+5 & ,\text{if } x\leq 2 \\ x-1 & ,\text{if } x=2 \end{cases}$

(iv) $f(x)=\begin{cases} k(x^2+3x) & ,\text{if } x\leq 0 \\ \cos 2x & ,\text{if } x\geq 0 \end{cases}$

(v) $f(x)=\begin{cases} \dfrac{\sqrt{1+px}-\sqrt{1-px}}{x} & ,\text{if } -1\leq x<0 \\ \dfrac{2x+1}{x-2} & ,\text{if } 0\leq x\leq 1 \end{cases}$

40. The function $f(x)=\begin{cases}\dfrac{x^2}{a} & , \text{if } 0\le x<1\\ a & , \text{if } 1\le x<\sqrt{2}\\ \dfrac{2b^2-4b}{x^2} & , \text{if } \sqrt{2}\le x<\infty\end{cases}$

is continuous on [0, ∞], then find the most suitable values of a and b.

41. Discuss the continuity of the function

$$f(x)=\begin{cases}2x-1, \text{if } x<2\\ \dfrac{3x}{2}, \text{if } x\ge 2\end{cases}$$

42. Prove that : $f(x)=\begin{cases}\dfrac{\sin x}{x}, \text{if } x<0\\ x+1, \text{if } x\ge 0\end{cases}$

43. Show that the function g(x) = x – [x] is discontinuous at all integral points. Here [x] denotes the greatest integer function.

44. Find all the points of discontinuity of f defined by f(x) = |x| – |x + 1|.

EXERCISE-2

1. The value of a for which the function

$$f(x)=\begin{cases}\dfrac{(4^x-1)^3}{\sin\left(\dfrac{x}{a}\right)\log\left[\left(1+\dfrac{x^2}{3}\right)\right]}, & x\ne 0\\ 12(\log 4)^3 & , x=0\end{cases}$$

(a) 1 (b) 2 (c) 3 (d) none of these

2. The function $f(x)=\begin{cases}\dfrac{\sin 3x}{x}, & x\ne 0\\ \dfrac{k}{2} & , x=0\end{cases}$ is continuous at x = 0, then k =

(a) 3 (b) 6 (c) 9 (d) 12

3. If $f(x)=\dfrac{1}{1-x}$, then the set of points discontinuity of the function f(f(f(x))) is

(a) {1} (b) {0, 1} (c) {–1, 1} (d) none of these

4. Let $f(x) = \dfrac{\tan\left(\frac{\pi}{4} - x\right)}{\cot 2x}, x \neq \frac{\pi}{4}$. The value which should be assigned to f(x) at $x = \frac{\pi}{4}$, so that it is continuous everywhere is

(a) 1 (b) $\frac{1}{2}$ (c) 2 (d) none of these

5. If the function f(x) defined by

$$f(x) = \begin{cases} \dfrac{\log(1+3x) - \log(1-2x)}{x} & , x \neq 0 \\ k & , x = 0 \end{cases}$$ is continuous at x = 0, then k =

(a) 1 (b) 5 (c) –1 (d) none of these

6. The values of the constants a, b and c for which the function

$$f(x) = \begin{cases} (a+ax)^{1/x} & , x < 0 \\ b & , x = 0 \\ \dfrac{(x+c)^{1/3} - 1}{(x+1)^{1/2} - 1} & , x > 0 \end{cases}$$ may be continuous at x = 0, are

(a) $a = \log_e\left(\frac{2}{3}\right), b = -\frac{2}{3}, c = 1$ (b) $a = \log_e\left(\frac{2}{3}\right), b = \frac{2}{3}, c = -1$

(c) $a = \log_e\left(\frac{2}{3}\right), b = \frac{2}{3}, c = 1$ (a) none of these

7. The points of discontinuity of the function

$$f(x) = \begin{cases} 2\sqrt{x} & , 0 \leq x \leq 1 \\ 4 - 2x & , 1 < x < \frac{5}{2} \\ 2x - 7 & , \frac{5}{2} \leq x \leq 4 \end{cases}$$ is (are)

(a) $x = 1, x = \frac{5}{2}$ (b) $x = \frac{5}{2}$ (c) $x = 1, \frac{5}{2}, 4$ (d) x = 0, 4

8. If $f(x) = \begin{cases} \dfrac{1-\sin^2 x}{3\cos^2 x} & , x < \dfrac{\pi}{2} \\ a & , x = \dfrac{\pi}{2} \\ \dfrac{b(1-\sin x)}{(\pi-2x)^2} & , x > \dfrac{\pi}{2} \end{cases}$. Then f(x) is continuous at $x = \dfrac{\pi}{2}$, if

(a) $a = \dfrac{1}{3}, b = 2$ (b) $a = \dfrac{1}{3}, b = \dfrac{8}{3}$ (c) $a = \dfrac{2}{3}, b = \dfrac{8}{3}$ (d) none of these

9. The points of discontinuity of the function

$$f(x) = \begin{cases} \frac{1}{5}(2x^2+3) & , x \le 1 \\ 6-5x & , 1 < x < 3 \\ x-3 & , x \ge 3 \end{cases}$$ is (are)

(a) $x = 1$ (b) $x = 3$ (c) $x = 1, 3$ (d) none of these

10. If $f(x) = \begin{cases} \dfrac{\sin(\cos x)-\cos x}{(\pi-2x)^2} & , x \ne \dfrac{\pi}{2} \\ k & , x = \dfrac{\pi}{2} \end{cases}$ is continuous at $x = \dfrac{\pi}{2}$, then k is equal to

(a) 0 (b) $\dfrac{1}{2}$ (c) 1 (d) –1