Exercise 11.1

Question 1:

If a line makes angles 90°, 135°, 45° with *x*, *y* and *z*-axes respectively, find its direction cosines.

Answer

Let direction cosines of the line be *l*, *m*, and *n*.

$$l = \cos 90° = 0$$
$$m = \cos 135° = -\frac{1}{\sqrt{2}}$$
$$n = \cos 45° = \frac{1}{\sqrt{2}}$$

Therefore, the direction cosines of the line are $0, -\frac{1}{\sqrt{2}}, \text{ and } \frac{1}{\sqrt{2}}.$

Question 2:

Find the direction cosines of a line which makes equal angles with the coordinate axes.

Answer

Let the direction cosines of the line make an angle *α* with each of the coordinate axes.

$\therefore$ $l = \cos\alpha$, $m = \cos\alpha$, $n = \cos\alpha$

$$l^2 + m^2 + n^2 = 1$$
$$\Rightarrow \cos^2\alpha + \cos^2\alpha + \cos^2\alpha = 1$$
$$\Rightarrow 3\cos^2\alpha = 1$$
$$\Rightarrow \cos^2\alpha = \frac{1}{3}$$
$$\Rightarrow \cos\alpha = \pm\frac{1}{\sqrt{3}}$$

Thus, the direction cosines of the line, which is equally inclined to the coordinate axes,

are $\pm\frac{1}{\sqrt{3}}, \pm\frac{1}{\sqrt{3}}, \text{ and } \pm\frac{1}{\sqrt{3}}$.

Question 3:

If a line has the direction ratios −18, 12, −4, then what are its direction cosines?

Answer

If a line has direction ratios of −18, 12, and −4, then its direction cosines are

$$\frac{-18}{\sqrt{(-18)^2+(12)^2+(-4)^2}}, \frac{12}{\sqrt{(-18)^2+(12)^2+(-4)^2}}, \frac{-4}{\sqrt{(-18)^2+(12)^2+(-4)^2}}$$

i.e., $\frac{-18}{22}, \frac{12}{22}, \frac{-4}{22}$

$\frac{-9}{11}, \frac{6}{11}, \frac{-2}{11}$

Thus, the direction cosines are $-\frac{9}{11}, \frac{6}{11}, \text{ and } \frac{-2}{11}$.

Question 4:

Show that the points (2, 3, 4), (−1, −2, 1), (5, 8, 7) are collinear.

Answer

The given points are A (2, 3, 4), B (− 1, − 2, 1), and C (5, 8, 7).

It is known that the direction ratios of line joining the points, (x_1, y_1, z_1) and (x_2, y_2, z_2), are given by, $x_2 - x_1$, $y_2 - y_1$, and $z_2 - z_1$.

The direction ratios of AB are (−1 − 2), (−2 − 3), and (1 − 4) i.e., −3, −5, and −3.

The direction ratios of BC are (5 − (− 1)), (8 − (− 2)), and (7 − 1) i.e., 6, 10, and 6.

It can be seen that the direction ratios of BC are −2 times that of AB i.e., they are proportional.

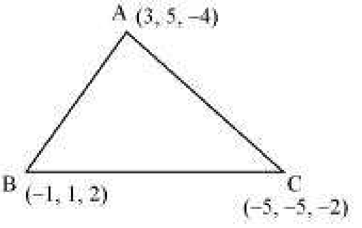
Therefore, AB is parallel to BC. Since point B is common to both AB and BC, points A, B, and C are collinear.

Question 5:

Find the direction cosines of the sides of the triangle whose vertices are (3, 5, − 4), (− 1, 1, 2) and (− 5, − 5, − 2)

Answer

The vertices of ΔABC are A (3, 5, −4), B (−1, 1, 2), and C (−5, −5, −2).



The direction ratios of side AB are (−1 − 3), (1 − 5), and (2 − (−4)) i.e., −4, −4, and 6.

$$\text{Then,} \sqrt{(-4)^2+(-4)^2+(6)^2} = \sqrt{16+16+36} = \sqrt{68} = 2\sqrt{17}$$

Therefore, the direction cosines of AB are

$$\frac{-4}{\sqrt{(-4)^2+(-4)^2+(6)^2}}, \frac{-4}{\sqrt{(-4)^2+(-4)^2+(6)^2}}, \frac{6}{\sqrt{(-4)^2+(-4)^2+(6)^2}}$$

$$\frac{-4}{2\sqrt{17}}, -\frac{4}{2\sqrt{17}}, \frac{6}{2\sqrt{17}}$$

$$\frac{-2}{\sqrt{17}}, \frac{-2}{\sqrt{17}}, \frac{3}{\sqrt{17}}$$

The direction ratios of BC are (−5 − (−1)), (−5 − 1), and (−2 − 2) i.e., −4, −6, and −4.

Therefore, the direction cosines of BC are

$$\frac{-4}{\sqrt{(-4)^2+(-6)^2+(-4)^2}}, \frac{-6}{\sqrt{(-4)^2+(-6)^2+(-4)^2}}, \frac{-4}{\sqrt{(-4)^2+(-6)^2+(-4)^2}}$$

i.e., $\frac{-4}{2\sqrt{17}}, \frac{-6}{2\sqrt{17}}, \frac{-4}{2\sqrt{17}}$

The direction ratios of CA are (−5 − 3), (−5 − 5), and (−2 − (−4)) i.e., −8, −10, and 2.

Therefore, the direction cosines of AC are

$$\frac{-8}{\sqrt{(-8)^2+(10)^2+(2)^2}}, \frac{-5}{\sqrt{(-8)^2+(10)^2+(2)^2}}, \frac{2}{\sqrt{(-8)^2+(10)^2+(2)^2}}$$

i.e., $\frac{-8}{2\sqrt{42}}, \frac{-10}{2\sqrt{42}}, \frac{2}{2\sqrt{42}}$

Exercise 11.2

Question 1:

Show that the three lines with direction cosines

$\frac{12}{13}, \frac{-3}{13}, \frac{-4}{13}; \frac{4}{13}, \frac{12}{13}, \frac{3}{13}; \frac{3}{13}, \frac{-4}{13}, \frac{12}{13}$ are mutually perpendicular.

Answer

Two lines with direction cosines, l_1, m_1, n_1 and l_2, m_2, n_2, are perpendicular to each other, if $l_1l_2 + m_1m_2 + n_1n_2 = 0$

(i) For the lines with direction cosines, $\frac{12}{13}, \frac{-3}{13}, \frac{-4}{13}$ and $\frac{4}{13}, \frac{12}{13}, \frac{3}{13}$, we obtain

$$l_1l_2 + m_1m_2 + n_1n_2 = \frac{12}{13} \times \frac{4}{13} + \left(\frac{-3}{13}\right) \times \frac{12}{13} + \left(\frac{-4}{13}\right) \times \frac{3}{13}$$
$$= \frac{48}{169} - \frac{36}{169} - \frac{12}{169}$$
$$= 0$$

Therefore, the lines are perpendicular.

(ii) For the lines with direction cosines, $\frac{4}{13}, \frac{12}{13}, \frac{3}{13}$ and $\frac{3}{13}, \frac{-4}{13}, \frac{12}{13}$, we obtain

$$l_1l_2 + m_1m_2 + n_1n_2 = \frac{4}{13} \times \frac{3}{13} + \frac{12}{13} \times \left(\frac{-4}{13}\right) + \frac{3}{13} \times \frac{12}{13}$$
$$= \frac{12}{169} - \frac{48}{169} + \frac{36}{169}$$
$$= 0$$

Therefore, the lines are perpendicular.

(iii) For the lines with direction cosines, $\frac{3}{13}, \frac{-4}{13}, \frac{12}{13}$ and $\frac{12}{13}, \frac{-3}{13}, \frac{-4}{13}$, we obtain

$$l_1l_2 + m_1m_2 + n_1n_2 = \left(\frac{3}{13}\right)\times\left(\frac{12}{13}\right)+\left(\frac{-4}{13}\right)\times\left(\frac{-3}{13}\right)+\left(\frac{12}{13}\right)\times\left(\frac{-4}{13}\right)$$
$$= \frac{36}{169}+\frac{12}{169}-\frac{48}{169}$$
$$= 0$$

Therefore, the lines are perpendicular.

Thus, all the lines are mutually perpendicular.

Question 2:

Show that the line through the points (1, −1, 2) (3, 4, −2) is perpendicular to the line through the points (0, 3, 2) and (3, 5, 6).

Answer

Let AB be the line joining the points, (1, −1, 2) and (3, 4, − 2), and CD be the line joining the points, (0, 3, 2) and (3, 5, 6).

The direction ratios, a_1, b_1, c_1, of AB are (3 − 1), (4 − (−1)), and (−2 − 2) i.e., 2, 5, and −4.

The direction ratios, a_2, b_2, c_2, of CD are (3 − 0), (5 − 3), and (6 −2) i.e., 3, 2, and 4.

AB and CD will be perpendicular to each other, if $a_1a_2 + b_1b_2 + c_1c_2 = 0$

$a_1a_2 + b_1b_2 + c_1c_2 = 2 \times 3 + 5 \times 2 + (-4) \times 4$

$= 6 + 10 - 16$

$= 0$

Therefore, AB and CD are perpendicular to each other.

Question 3:

Show that the line through the points (4, 7, 8) (2, 3, 4) is parallel to the line through the points (−1, −2, 1), (1, 2, 5).

Answer

Let AB be the line through the points, (4, 7, 8) and (2, 3, 4), and CD be the line through the points, (−1, −2, 1) and (1, 2, 5).

The directions ratios, a_1, b_1, c_1, of AB are (2 − 4), (3 − 7), and (4 − 8) i.e., −2, −4, and −4.

The direction ratios, a_2, b_2, c_2, of CD are (1 − (−1)), (2 − (−2)), and (5 − 1) i.e., 2, 4, and 4.

AB will be parallel to CD, if $\dfrac{a_1}{a_2}=\dfrac{b_1}{b_2}=\dfrac{c_1}{c_2}$

$$\frac{a_1}{a_2}=\frac{-2}{2}=-1$$

$$\frac{b_1}{b_2}=\frac{-4}{4}=-1$$

$$\frac{c_1}{c_2}=\frac{-4}{4}=-1$$

$$\therefore\ \frac{a_1}{a_2}=\frac{b_1}{b_2}=\frac{c_1}{c_2}$$

Thus, AB is parallel to CD.

Question 4:

Find the equation of the line which passes through the point (1, 2, 3) and is parallel to the vector $3\hat{i}+2\hat{j}-2\hat{k}$.

Answer

It is given that the line passes through the point A (1, 2, 3). Therefore, the position vector through A is $\vec{a}=\hat{i}+2\hat{j}+3\hat{k}$

$\vec{b}=3\hat{i}+2\hat{j}-2\hat{k}$

It is known that the line which passes through point A and parallel to $\vec{b}$ is given by

$\vec{r}=\vec{a}+\lambda\vec{b}$, where λ is a constant.

$\Rightarrow \vec{r}=\hat{i}+2\hat{j}+3\hat{k}+\lambda\left(3\hat{i}+2\hat{j}-2\hat{k}\right)$

This is the required equation of the line.

Question 5:

Find the equation of the line in vector and in Cartesian form that passes through the point with position vector $2\hat{i}-\hat{j}+4\hat{k}$ and is in the direction $\hat{i}+2\hat{j}-\hat{k}$.

Answer

It is given that the line passes through the point with position vector

$\vec{a} = 2\hat{i} - \hat{j} + 4\hat{k}$ ...(1)

$\vec{b} = \hat{i} + 2\hat{j} - \hat{k}$ ...(2)

It is known that a line through a point with position vector $\vec{a}$ and parallel to $\vec{b}$ is given by the equation, $\vec{r} = \vec{a} + \lambda\vec{b}$

$\Rightarrow \vec{r} = 2\hat{i} - \hat{j} + 4\hat{k} + \lambda\left(\hat{i} + 2\hat{j} - \hat{k}\right)$

This is the required equation of the line in vector form.

$\vec{r} = x\hat{i} - y\hat{j} + z\hat{k}$

$\Rightarrow x\hat{i} - y\hat{j} + z\hat{k} = (\lambda + 2)\hat{i} + (2\lambda - 1)\hat{j} + (-\lambda + 4)\hat{k}$

Eliminating λ, we obtain the Cartesian form equation as

$$\frac{x-2}{1} = \frac{y+1}{2} = \frac{z-4}{-1}$$

This is the required equation of the given line in Cartesian form.

Question 6:

Find the Cartesian equation of the line which passes through the point

(−2, 4, −5) and parallel to the line given by $\frac{x+3}{3} = \frac{y-4}{5} = \frac{z+8}{6}$

Answer

It is given that the line passes through the point (−2, 4, −5) and is parallel to

$$\frac{x+3}{3} = \frac{y-4}{5} = \frac{z+8}{6}$$

The direction ratios of the line, $\frac{x+3}{3} = \frac{y-4}{5} = \frac{z+8}{6}$, are 3, 5, and 6.

The required line is parallel to $\frac{x+3}{3} = \frac{y-4}{5} = \frac{z+8}{6}$

Therefore, its direction ratios are $3k$, $5k$, and $6k$, where $k \neq 0$

It is known that the equation of the line through the point (x_1, y_1, z_1) and with direction

ratios, a, b, c, is given by $\frac{x-x_1}{a} = \frac{y-y_1}{b} = \frac{z-z_1}{c}$

Therefore the equation of the required line is

$$\frac{x+2}{3k}=\frac{y-4}{5k}=\frac{z+5}{6k}$$

$$\Rightarrow \frac{x+2}{3}=\frac{y-4}{5}=\frac{z+5}{6}=k$$

Question 7:

The Cartesian equation of a line is $\frac{x-5}{3}=\frac{y+4}{7}=\frac{z-6}{2}$. Write its vector form.

Answer

The Cartesian equation of the line is

$$\frac{x-5}{3}=\frac{y+4}{7}=\frac{z-6}{2} \quad \dots (1)$$

The given line passes through the point (5, −4, 6). The position vector of this point is

$\vec{a}=5\hat{i}-4\hat{j}+6\hat{k}$

Also, the direction ratios of the given line are 3, 7, and 2.

This means that the line is in the direction of vector, $\vec{b}=3\hat{i}+7\hat{j}+2\hat{k}$

It is known that the line through position vector $\vec{a}$ and in the direction of the vector $\vec{b}$ is given by the equation, $\vec{r}=\vec{a}+\lambda\vec{b}, \lambda \in R$

$$\Rightarrow \vec{r}=\left(5\hat{i}-4\hat{j}+6\hat{k}\right)+\lambda\left(3\hat{i}+7\hat{j}+2\hat{k}\right)$$

This is the required equation of the given line in vector form.

Question 8:

Find the vector and the Cartesian equations of the lines that pass through the origin and (5, −2, 3).

Answer

The required line passes through the origin. Therefore, its position vector is given by,

$$\vec{a}=\vec{0} \quad \dots (1)$$

The direction ratios of the line through origin and (5, −2, 3) are

(5 − 0) = 5, (−2 − 0) = −2, (3 − 0) = 3

The line is parallel to the vector given by the equation, $\vec{b} = 5\hat{i} - 2\hat{j} + 3\hat{k}$

The equation of the line in vector form through a point with position vector $\vec{a}$ and parallel to $\vec{b}$ is, $\vec{r} = \vec{a} + \lambda\vec{b},\ \lambda \in R$

$\Rightarrow \vec{r} = \vec{0} + \lambda\left(5\hat{i} - 2\hat{j} + 3\hat{k}\right)$

$\Rightarrow \vec{r} = \lambda\left(5\hat{i} - 2\hat{j} + 3\hat{k}\right)$

The equation of the line through the point (x_1, y_1, z_1) and direction ratios *a*, *b*, *c* is given by, $\frac{x - x_1}{a} = \frac{y - y_1}{b} = \frac{z - z_1}{c}$

Therefore, the equation of the required line in the Cartesian form is

$\frac{x - 0}{5} = \frac{y - 0}{-2} = \frac{z - 0}{3}$

$\Rightarrow \frac{x}{5} = \frac{y}{-2} = \frac{z}{3}$

Question 9:

Find the vector and the Cartesian equations of the line that passes through the points (3, −2, −5), (3, −2, 6).

Answer

Let the line passing through the points, P (3, −2, −5) and Q (3, −2, 6), be PQ.

Since PQ passes through P (3, −2, −5), its position vector is given by,

$\vec{a} = 3\hat{i} - 2\hat{j} - 5\hat{k}$

The direction ratios of PQ are given by,

(3 − 3) = 0, (−2 + 2) = 0, (6 + 5) = 11

The equation of the vector in the direction of PQ is

$\vec{b} = 0.\hat{i} - 0.\hat{j} + 11\hat{k} = 11\hat{k}$

The equation of PQ in vector form is given by, $\vec{r} = \vec{a} + \lambda\vec{b},\ \lambda \in R$

$\Rightarrow \vec{r} = \left(3\hat{i} - 2\hat{j} - 5\hat{k}\right) + 11\lambda\hat{k}$

The equation of PQ in Cartesian form is

$$\frac{x-x_1}{a}=\frac{y-y_1}{b}=\frac{z-z_1}{c} \text{ i.e., } \frac{x-3}{0}=\frac{y+2}{0}=\frac{z+5}{11}$$

Question 10:

Find the angle between the following pairs of lines:

(i) $\vec{r}=2\hat{i}-5\hat{j}+\hat{k}+\lambda\left(3\hat{i}-2\hat{j}+6\hat{k}\right)$ and

$\vec{r}=7\hat{i}-6\hat{k}+\mu\left(\hat{i}+2\hat{j}+2\hat{k}\right)$

(ii) $\vec{r}=3\hat{i}+\hat{j}-2\hat{k}+\lambda\left(\hat{i}-\hat{j}-2\hat{k}\right)$ and

$\vec{r}=2\hat{i}-\hat{j}-56\hat{k}+\mu\left(3\hat{i}-5\hat{j}-4\hat{k}\right)$

Answer

(i) Let Q be the angle between the given lines.

The angle between the given pairs of lines is given by, $\cos Q=\left|\frac{\vec{b}_1\cdot\vec{b}_2}{\left|\vec{b}_1\right|\left|\vec{b}_2\right|}\right|$

The given lines are parallel to the vectors, $\vec{b}_1=3\hat{i}+2\hat{j}+6\hat{k}$ and $\vec{b}_2=\hat{i}+2\hat{j}+2\hat{k}$, respectively.

$\therefore\left|\vec{b}_1\right|=\sqrt{3^2+2^2+6^2}=7$

$\left|\vec{b}_2\right|=\sqrt{(1)^2+(2)^2+(2)^2}=3$

$$\begin{aligned}\vec{b}_1\cdot\vec{b}_2&=\left(3\hat{i}+2\hat{j}+6\hat{k}\right)\cdot\left(\hat{i}+2\hat{j}+2\hat{k}\right)\\&=3\times1+2\times2+6\times2\\&=3+4+12\\&=19\end{aligned}$$

$$\Rightarrow \cos Q = \frac{19}{7\times 3}$$

$$\Rightarrow Q = \cos^{-1}\left(\frac{19}{21}\right)$$

(ii) The given lines are parallel to the vectors, $\vec{b}_1 = \hat{i} - \hat{j} - 2\hat{k}$ and $\vec{b}_2 = 3\hat{i} - 5\hat{j} - 4\hat{k}$, respectively.

$$\therefore \left|\vec{b}_1\right| = \sqrt{(1)^2 + (-1)^2 + (-2)^2} = \sqrt{6}$$

$$\left|\vec{b}_2\right| = \sqrt{(3)^2 + (-5)^2 + (-4)^2} = \sqrt{50} = 5\sqrt{2}$$

$$\vec{b}_1 \cdot \vec{b}_2 = \left(\hat{i} - \hat{j} - 2\hat{k}\right)\cdot\left(3\hat{i} - 5\hat{j} - 4\hat{k}\right)$$

$$= 1\cdot 3 - 1(-5) - 2(-4)$$

$$= 3 + 5 + 8$$

$$= 16$$

$$\cos Q = \left|\frac{\vec{b}_1 \cdot \vec{b}_2}{\left|\vec{b}_1\right|\left|\vec{b}_2\right|}\right|$$

$$\Rightarrow \cos Q = \frac{16}{\sqrt{6}\cdot 5\sqrt{2}} = \frac{16}{\sqrt{2}\cdot\sqrt{3}\cdot 5\sqrt{2}} = \frac{16}{10\sqrt{3}}$$

$$\Rightarrow \cos Q = \frac{8}{5\sqrt{3}}$$

$$\Rightarrow Q = \cos^{-1}\left(\frac{8}{5\sqrt{3}}\right)$$

Question 11:

Find the angle between the following pairs of lines:

(i) $\frac{x-2}{2} = \frac{y-1}{5} = \frac{z+3}{-3}$ and $\frac{x+2}{-1} = \frac{y-4}{8} = \frac{z-5}{4}$

(ii) $\frac{x}{2} = \frac{y}{2} = \frac{z}{1}$ and $\frac{x-5}{4} = \frac{y-2}{1} = \frac{z-3}{8}$

Answer

Let $\vec{b}_1$ and $\vec{b}_2$ be the vectors parallel to the pair of lines,

$\frac{x-2}{2}=\frac{y-1}{5}=\frac{z+3}{-3}$ and $\frac{x+2}{-1}=\frac{y-4}{8}=\frac{z-5}{4}$, respectively.

$\therefore \vec{b}_1 = 2\hat{i}+5\hat{j}-3\hat{k}$ and $\vec{b}_2 = -\hat{i}+8\hat{j}+4\hat{k}$

$$\left|\vec{b}_1\right| = \sqrt{(2)^2+(5)^2+(-3)^2} = \sqrt{38}$$

$$\left|\vec{b}_2\right| = \sqrt{(-1)^2+(8)^2+(4)^2} = \sqrt{81} = 9$$

$$\begin{aligned}\vec{b}_1\cdot\vec{b}_2 &= \left(2\hat{i}+5\hat{j}-3\hat{k}\right)\cdot\left(-\hat{i}+8\hat{j}+4\hat{k}\right)\\ &= 2(-1)+5\times 8+(-3)\cdot 4\\ &= -2+40-12\\ &= 26\end{aligned}$$

The angle, Q, between the given pair of lines is given by the relation,

$$\cos Q = \left|\frac{\vec{b}_1\cdot\vec{b}_2}{\left|\vec{b}_1\right|\left|\vec{b}_2\right|}\right|$$

$$\Rightarrow \cos Q = \frac{26}{9\sqrt{38}}$$

$$\Rightarrow Q = \cos^{-1}\left(\frac{26}{9\sqrt{38}}\right)$$

(ii) Let $\vec{b}_1, \vec{b}_2$ be the vectors parallel to the given pair of lines, $\frac{x}{2}=\frac{y}{2}=\frac{z}{1}$ and

$\frac{x-5}{4}=\frac{y-5}{1}=\frac{z-3}{8}$, respectively.

$\vec{b}_1 = 2\hat{i} + 2\hat{j} + \hat{k}$

$\vec{b}_2 = 4\hat{i} + \hat{j} + 8\hat{k}$

$\therefore \left|\vec{b}_1\right| = \sqrt{(2)^2 + (2)^2 + (1)^2} = \sqrt{9} = 3$

$\left|\vec{b}_2\right| = \sqrt{4^2 + 1^2 + 8^2} = \sqrt{81} = 9$

$$\begin{aligned}\vec{b}_1 \cdot \vec{b}_2 &= \left(2\hat{i} + 2\hat{j} + \hat{k}\right)\cdot\left(4\hat{i} + \hat{j} + 8\hat{k}\right)\\ &= 2\times4 + 2\times1 + 1\times8\\ &= 8 + 2 + 8\\ &= 18\end{aligned}$$

If Q is the angle between the given pair of lines, then $\cos Q = \left|\dfrac{\vec{b}_1 \cdot \vec{b}_2}{\left|\vec{b}_1\right|\left|\vec{b}_2\right|}\right|$

$\Rightarrow \cos Q = \dfrac{18}{3\times9} = \dfrac{2}{3}$

$\Rightarrow Q = \cos^{-1}\left(\dfrac{2}{3}\right)$

Question 12:

Find the values of *p* so the line $\dfrac{1-x}{3} = \dfrac{7y-14}{2p} = \dfrac{z-3}{2}$ and $\dfrac{7-7x}{3p} = \dfrac{y-5}{1} = \dfrac{6-z}{5}$ are at right angles.

Answer

The given equations can be written in the standard form as

$\dfrac{x-1}{-3} = \dfrac{y-2}{\frac{2p}{7}} = \dfrac{z-3}{2}$ and $\dfrac{x-1}{\frac{-3p}{7}} = \dfrac{y-5}{1} = \dfrac{z-6}{-5}$

The direction ratios of the lines are $-3, \dfrac{2p}{7}$, 2 and $\dfrac{-3p}{7}, 1, -5$ respectively.

Two lines with direction ratios, a_1, b_1, c_1 and a_2, b_2, c_2, are perpendicular to each other, if $a_1a_2 + b_1 b_2 + c_1c_2 = 0$

$$\therefore (-3)\cdot\left(\frac{-3p}{7}\right)+\left(\frac{2p}{7}\right)\cdot(1)+2\cdot(-5)=0$$

$$\Rightarrow \frac{9p}{7}+\frac{2p}{7}=10$$

$$\Rightarrow 11p=70$$

$$\Rightarrow p=\frac{70}{11}$$

Thus, the value of p is $\frac{70}{11}$.

Question 13:

Show that the lines $\frac{x-5}{7}=\frac{y+2}{-5}=\frac{z}{1}$ and $\frac{x}{1}=\frac{y}{2}=\frac{z}{3}$ are perpendicular to each other.

Answer

The equations of the given lines are $\frac{x-5}{7}=\frac{y+2}{-5}=\frac{z}{1}$ and $\frac{x}{1}=\frac{y}{2}=\frac{z}{3}$

The direction ratios of the given lines are 7, −5, 1 and 1, 2, 3 respectively.

Two lines with direction ratios, a_1, b_1, c_1 and a_2, b_2, c_2, are perpendicular to each other, if $a_1a_2 + b_1 b_2 + c_1c_2 = 0$

$\therefore$ 7 × 1 + (−5) × 2 + 1 × 3

= 7 − 10 + 3

= 0

Therefore, the given lines are perpendicular to each other.

Question 14:

Find the shortest distance between the lines

$$\vec{r}=\left(\hat{i}+2\hat{j}+\hat{k}\right)+\lambda\left(\hat{i}-\hat{j}+\hat{k}\right)\text{and}$$

$$\vec{r}=2\hat{i}-\hat{j}-\hat{k}+\mu\left(2\hat{i}+\hat{j}+2\hat{k}\right)$$

Answer

The equations of the given lines are

$$\vec{r} = \left(\hat{i} + 2\hat{j} + \hat{k}\right) + \lambda\left(\hat{i} - \hat{j} + \hat{k}\right)$$

$$\vec{r} = 2\hat{i} - \hat{j} - \hat{k} + \mu\left(2\hat{i} + \hat{j} + 2\hat{k}\right)$$

It is known that the shortest distance between the lines, $\vec{r} = \vec{a}_1 + \lambda\vec{b}_1$ and $\vec{r} = \vec{a}_2 + \mu\vec{b}_2$, is given by,

$$d = \left|\frac{\left(\vec{b}_1 \times \vec{b}_2\right)\cdot\left(\vec{a}_2 - \vec{a}_2\right)}{\left|\vec{b}_1 \times \vec{b}_2\right|}\right| \qquad \ldots(1)$$

Comparing the given equations, we obtain

$$\vec{a}_1 = \hat{i} + 2\hat{j} + \hat{k}$$

$$\vec{b}_1 = \hat{i} - \hat{j} + \hat{k}$$

$$\vec{a}_2 = 2\hat{i} - \hat{j} - \hat{k}$$

$$\vec{b}_2 = 2\hat{i} + \hat{j} + 2\hat{k}$$

$$\vec{a}_2 - \vec{a}_1 = \left(2\hat{i} - \hat{j} - \hat{k}\right) - \left(\hat{i} + 2\hat{j} + \hat{k}\right) = \hat{i} - 3\hat{j} - 2\hat{k}$$

$$\vec{b}_1 \times \vec{b}_2 = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & -1 & 1 \\ 2 & 1 & 2 \end{vmatrix}$$

$$\vec{b}_1 \times \vec{b}_2 = (-2-1)\hat{i} - (2-2)\hat{j} + (1+2)\hat{k} = -3\hat{i} + 3\hat{k}$$

$$\Rightarrow \left|\vec{b}_1 \times \vec{b}_2\right| = \sqrt{(-3)^2 + (3)^2} = \sqrt{9+9} = \sqrt{18} = 3\sqrt{2}$$

Substituting all the values in equation (1), we obtain

$$d = \left| \frac{\left(-3\hat{i} + 3\hat{k}\right) \cdot \left(\hat{i} - 3\hat{j} - 2\hat{k}\right)}{3\sqrt{2}} \right|$$

$$\Rightarrow d = \left| \frac{-3.1 + 3(-2)}{3\sqrt{2}} \right|$$

$$\Rightarrow d = \left| \frac{-9}{3\sqrt{2}} \right|$$

$$\Rightarrow d = \frac{3}{\sqrt{2}} = \frac{3 \times \sqrt{2}}{\sqrt{2} \times \sqrt{2}} = \frac{3\sqrt{2}}{2}$$

Therefore, the shortest distance between the two lines is $\frac{3\sqrt{2}}{2}$ units.

Question 15:

Find the shortest distance between the lines $\frac{x+1}{7} = \frac{y+1}{-6} = \frac{z+1}{1}$ and $\frac{x-3}{1} = \frac{y-5}{-2} = \frac{z-7}{1}$

Answer

The given lines are $\frac{x+1}{7} = \frac{y+1}{-6} = \frac{z+1}{1}$ and $\frac{x-3}{1} = \frac{y-5}{-2} = \frac{z-7}{1}$

It is known that the shortest distance between the two lines,

$\frac{x-x_1}{a_1} = \frac{y-y_1}{b_1} = \frac{z-z_1}{c_1}$ and $\frac{x-x_2}{a_2} = \frac{y-y_2}{b_2} = \frac{z-z_2}{c_2}$**,** is given by,

$$d = \frac{\begin{vmatrix} x_2 - x_1 & y_2 - y_1 & z_2 - z_1 \\ a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{vmatrix}}{\sqrt{(b_1c_2 - b_2c_1)^2 + (c_1a_2 - c_2a_1)^2 + (a_1b_2 - a_2b_1)^2}} \qquad \ldots(1)$$

Comparing the given equations, we obtain

$$x_1=-1,\ y_1=-1,\ z_1=-1$$
$$a_1=7,\ \ b_1=-6,\ c_1=1$$
$$x_2=3,\ \ y_2=5,\ z_2=7$$
$$a_2=1,\ \ b_2=-2,\ c_2=1$$

$$\text{Then,}\begin{vmatrix} x_2-x_1 & y_2-y_1 & z_2-z_1 \\ a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{vmatrix}=\begin{vmatrix} 4 & 6 & 8 \\ 7 & -6 & 1 \\ 1 & -2 & 1 \end{vmatrix}$$
$$=4(-6+2)-6(7-1)+8(-14+6)$$
$$=-16-36-64$$
$$=-116$$

$$\Rightarrow \sqrt{(b_1c_2-b_2c_1)^2+(c_1a_2-c_2a_1)^2+(a_1b_2-a_2b_1)^2}=\sqrt{(-6+2)^2+(1+7)^2+(-14+6)^2}$$
$$=\sqrt{16+36+64}$$
$$=\sqrt{116}$$
$$=2\sqrt{29}$$

Substituting all the values in equation (1), we obtain

$$d=\frac{-116}{2\sqrt{29}}=\frac{-58}{\sqrt{29}}=\frac{-2\times 29}{\sqrt{29}}=-2\sqrt{29}$$

Since distance is always non-negative, the distance between the given lines is $2\sqrt{29}$ units.

Question 16:

Find the shortest distance between the lines whose vector equations are

$$\vec{r}=\left(\hat{i}+2\hat{j}+3\hat{k}\right)+\lambda\left(\hat{i}-3\hat{j}+2\hat{k}\right)$$
$$\text{and}\ \vec{r}=4\hat{i}+5\hat{j}+6\hat{k}+\mu\left(2\hat{i}+3\hat{j}+\hat{k}\right)$$

Answer

The given lines are $\vec{r}=\hat{i}+2\hat{j}+3\hat{k}+\lambda\left(\hat{i}-3\hat{j}+2\hat{k}\right)$ and $\vec{r}=4\hat{i}+5\hat{j}+6\hat{k}+\mu\left(2\hat{i}+3\hat{j}+\hat{k}\right)$

It is known that the shortest distance between the lines, $\vec{r}=\vec{a}_1+\lambda\vec{b}_1$ and $\vec{r}=\vec{a}_2+\mu\vec{b}_2$, is given by,

$$d = \left| \frac{\left(\vec{b}_1 \times \vec{b}_2\right) \cdot \left(\vec{a}_2 - \vec{a}_2\right)}{\left|\vec{b}_1 \times \vec{b}_2\right|} \right| \qquad \ldots(1)$$

Comparing the given equations with $\vec{r} = \vec{a}_1 + \lambda \vec{b}_1$ and $\vec{r} = \vec{a}_2 + \mu \vec{b}_2$, we obtain

$\vec{a}_1 = \hat{i} + 2\hat{j} + 3\hat{k}$

$\vec{b}_1 = \hat{i} - 3\hat{j} + 2\hat{k}$

$\vec{a}_2 = 4\hat{i} + 5\hat{j} + 6\hat{k}$

$\vec{b}_2 = 2\hat{i} + 3\hat{j} + \hat{k}$

$\vec{a}_2 - \vec{a}_1 = \left(4\hat{i} + 5\hat{j} + 6\hat{k}\right) - \left(\hat{i} + 2\hat{j} + 3\hat{k}\right) = 3\hat{i} + 3\hat{j} + 3\hat{k}$

$$\vec{b}_1 \times \vec{b}_2 = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & -3 & 2 \\ 2 & 3 & 1 \end{vmatrix} = (-3-6)\hat{i} - (1-4)\hat{j} + (3+6)\hat{k} = -9\hat{i} + 3\hat{j} + 9\hat{k}$$

$$\Rightarrow \left|\vec{b}_1 \times \vec{b}_2\right| = \sqrt{(-9)^2 + (3)^2 + (9)^2} = \sqrt{81+9+81} = \sqrt{171} = 3\sqrt{19}$$

$$\begin{aligned} \left(\vec{b}_1 \times \vec{b}_2\right) \cdot \left(\vec{a}_2 - \vec{a}_1\right) &= \left(-9\hat{i} + 3\hat{j} + 9\hat{k}\right) \cdot \left(3\hat{i} + 3\hat{j} + 3\hat{k}\right) \\ &= -9 \times 3 + 3 \times 3 + 9 \times 3 \\ &= 9 \end{aligned}$$

Substituting all the values in equation (1), we obtain

$$d = \left| \frac{9}{3\sqrt{19}} \right| = \frac{3}{\sqrt{19}}$$

Therefore, the shortest distance between the two given lines is $\frac{3}{\sqrt{19}}$ units.

Question 17:

Find the shortest distance between the lines whose vector equations are

$\vec{r} = (1-t)\hat{i} + (t-2)\hat{j} + (3-2t)\hat{k}$ and

$\vec{r} = (s+1)\hat{i} + (2s-1)\hat{j} - (2s+1)\hat{k}$

Answer

The given lines are

$$\vec{r}=(1-t)\hat{i}+(t-2)\hat{j}+(3-2t)\hat{k}$$
$$\Rightarrow \vec{r}=\left(\hat{i}-2\hat{j}+3\hat{k}\right)+t\left(-\hat{i}+\hat{j}-2\hat{k}\right) \qquad ...(1)$$

$$\vec{r}=(s+1)\hat{i}+(2s-1)\hat{j}-(2s+1)\hat{k}$$
$$\Rightarrow \vec{r}=\left(\hat{i}-\hat{j}+\hat{k}\right)+s\left(\hat{i}+2\hat{j}-2\hat{k}\right) \qquad ...(2)$$

It is known that the shortest distance between the lines, $\vec{r}=\vec{a}_1+\lambda\vec{b}_1$ and $\vec{r}=\vec{a}_2+\mu\vec{b}_2$, is given by,

$$d=\left|\frac{\left(\vec{b}_1\times\vec{b}_2\right)\cdot\left(\vec{a}_2-\vec{a}_2\right)}{\left|\vec{b}_1\times\vec{b}_2\right|}\right| \qquad ...(3)$$

For the given equations,

$$\vec{a}_1=\hat{i}-2\hat{j}+3\hat{k}$$
$$\vec{b}_1=-\hat{i}+\hat{j}-2\hat{k}$$
$$\vec{a}_2=\hat{i}-\hat{j}-\hat{k}$$
$$\vec{b}_2=\hat{i}+2\hat{j}-2\hat{k}$$

$$\vec{a}_2-\vec{a}_1=\left(\hat{i}-\hat{j}-\hat{k}\right)-\left(\hat{i}-2\hat{j}+3\hat{k}\right)=\hat{j}-4\hat{k}$$

$$\vec{b}_1\times\vec{b}_2=\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ -1 & 1 & -2\\ 1 & 2 & -2\end{vmatrix}=(-2+4)\hat{i}-(2+2)\hat{j}+(-2-1)\hat{k}=2\hat{i}-4\hat{j}-3\hat{k}$$

$$\Rightarrow \left|\vec{b}_1\times\vec{b}_2\right|=\sqrt{(2)^2+(-4)^2+(-3)^2}=\sqrt{4+16+9}=\sqrt{29}$$

$$\therefore \left(\vec{b}_1\times\vec{b}_2\right)\cdot\left(\vec{a}_2-\vec{a}_1\right)=\left(2\hat{i}-4\hat{j}-3\hat{k}\right)\cdot\left(\hat{j}-4\hat{k}\right)=-4+12=8$$

Substituting all the values in equation (3), we obtain

$$d=\left|\frac{8}{\sqrt{29}}\right|=\frac{8}{\sqrt{29}}$$

Therefore, the shortest distance between the lines is $\frac{8}{\sqrt{29}}$ units.

Exercise 11.3

Question 1:

In each of the following cases, determine the direction cosines of the normal to the plane and the distance from the origin.

(a)z = 2 (b) $x+y+z=1$

(c) $2x+3y-z=5$ (d)5y + 8 = 0

Answer

(a) The equation of the plane is z = 2 or 0x + 0y + z = 2 … (1)

The direction ratios of normal are 0, 0, and 1.

$$\therefore \sqrt{0^2+0^2+1^2}=1$$

Dividing both sides of equation (1) by 1, we obtain

$$0.x+0.y+1.z=2$$

This is of the form $lx + my + nz = d$, where l, m, n are the direction cosines of normal to the plane and d is the distance of the perpendicular drawn from the origin.

Therefore, the direction cosines are 0, 0, and 1 and the distance of the plane from the origin is 2 units.

(b) $x + y + z = 1$ … (1)

The direction ratios of normal are 1, 1, and 1.

$$\therefore \sqrt{(1)^2+(1)^2+(1)^2}=\sqrt{3}$$

Dividing both sides of equation (1) by $\sqrt{3}$, we obtain

$$\frac{1}{\sqrt{3}}x+\frac{1}{\sqrt{3}}y+\frac{1}{\sqrt{3}}z=\frac{1}{\sqrt{3}} \qquad ...(2)$$

This equation is of the form $lx + my + nz = d$, where l, m, n are the direction cosines of normal to the plane and d is the distance of normal from the origin.

Therefore, the direction cosines of the normal are $\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \text{ and } \frac{1}{\sqrt{3}}$ and the distance of normal from the origin is $\frac{1}{\sqrt{3}}$ units.

(c) $2x + 3y - z = 5$ … (1)

The direction ratios of normal are 2, 3, and −1.

$$\therefore \sqrt{(2)^2 + (3)^2 + (-1)^2} = \sqrt{14}$$

Dividing both sides of equation (1) by $\sqrt{14}$, we obtain

$$\frac{2}{\sqrt{14}}x + \frac{3}{\sqrt{14}}y - \frac{1}{\sqrt{14}}z = \frac{5}{\sqrt{14}}$$

This equation is of the form $lx + my + nz = d$, where l, m, n are the direction cosines of normal to the plane and d is the distance of normal from the origin.

Therefore, the direction cosines of the normal to the plane are $\frac{2}{\sqrt{14}}, \frac{3}{\sqrt{14}}, \text{ and } \frac{-1}{\sqrt{14}}$ and

the distance of normal from the origin is $\frac{5}{\sqrt{14}}$ units.

(d) $5y + 8 = 0$

$\Rightarrow 0x - 5y + 0z = 8$ … (1)

The direction ratios of normal are 0, −5, and 0.

$$\therefore \sqrt{0 + (-5)^2 + 0} = 5$$

Dividing both sides of equation (1) by 5, we obtain

$$-y = \frac{8}{5}$$

This equation is of the form *lx* + *my* + *nz* = *d*, where *l*, *m*, *n* are the direction cosines of normal to the plane and *d* is the distance of normal from the origin.

Therefore, the direction cosines of the normal to the plane are 0, −1, and 0 and the distance of normal from the origin is $\frac{8}{5}$ units.

Question 2:

Find the vector equation of a plane which is at a distance of 7 units from the origin and normal to the vector $3\hat{i}+5\hat{j}-6\hat{k}$.

Answer

The normal vector is, $\vec{n}=3\hat{i}+5\hat{j}-6\hat{k}$

$$\therefore \hat{n}=\frac{\vec{n}}{|\vec{n}|}=\frac{3\hat{i}+5\hat{j}-6\hat{k}}{\sqrt{(3)^2+(5)^2+(6)^2}}=\frac{3\hat{i}+5\hat{j}-6\hat{k}}{\sqrt{70}}$$

It is known that the equation of the plane with position vector $\vec{r}$ is given by, $\vec{r}\cdot\hat{n}=d$

$$\Rightarrow \vec{r}\cdot\left(\frac{3\hat{i}+5\hat{j}-6\hat{k}}{\sqrt{70}}\right)=7$$

This is the vector equation of the required plane.

Question 3:

Find the Cartesian equation of the following planes:

(a) $\vec{r}\cdot\left(\hat{i}+\hat{j}-\hat{k}\right)=2$ (b) $\vec{r}\cdot\left(2\hat{i}+3\hat{j}-4\hat{k}\right)=1$

(c) $\vec{r}\cdot\left[(s-2t)\hat{i}+(3-t)\hat{j}+(2s+t)\hat{k}\right]=15$

Answer

(a) It is given that equation of the plane is

$$\vec{r}\cdot\left(\hat{i}+\hat{j}-\hat{k}\right)=2 \qquad ...(1)$$

For any arbitrary point P (*x*, *y*, *z*) on the plane, position vector $\vec{r}$ is given by,

$$\vec{r}=x\hat{i}+y\hat{j}-z\hat{k}$$

Substituting the value of $\vec{r}$ in equation (1), we obtain

$$\left(x\hat{i}+y\hat{j}-z\hat{k}\right)\cdot\left(\hat{i}+\hat{j}-\hat{k}\right)=2$$
$$\Rightarrow x+y-z=2$$

This is the Cartesian equation of the plane.

(b) $\vec{r}\cdot\left(2\hat{i}+3\hat{j}-4\hat{k}\right)=1 \qquad \ldots(1)$

For any arbitrary point P (*x*, *y*, *z*) on the plane, position vector $\vec{r}$ is given by,

$$\vec{r}=x\hat{i}+y\hat{j}-z\hat{k}$$

Substituting the value of $\vec{r}$ in equation (1), we obtain

$$\left(x\hat{i}+y\hat{j}+z\hat{k}\right)\cdot\left(2\hat{i}+3\hat{j}-4\hat{k}\right)=1$$
$$\Rightarrow 2x+3y-4z=1$$

This is the Cartesian equation of the plane.

(c) $\vec{r}\cdot\left[(s-2t)\hat{i}+(3-t)\hat{j}+(2s+t)\hat{k}\right]=15 \qquad \ldots(1)$

For any arbitrary point P (*x*, *y*, *z*) on the plane, position vector $\vec{r}$ is given by,

$$\vec{r}=x\hat{i}+y\hat{j}-z\hat{k}$$

Substituting the value of $\vec{r}$ in equation (1), we obtain

$$\left(x\hat{i}+y\hat{j}-z\hat{k}\right)\cdot\left[(s-2t)\hat{i}+(3-t)\hat{j}+(2s+t)\hat{k}\right]=15$$
$$\Rightarrow (s-2t)x+(3-t)y+(2s+t)z=15$$

This is the Cartesian equation of the given plane.

Question 4:

In the following cases, find the coordinates of the foot of the perpendicular drawn from the origin.

(a) $2x+3y+4z-12=0$ **(b)** $3y+4z-6=0$

(c) $x+y+z=1$ **(d)** $5y+8=0$

Answer

(a) Let the coordinates of the foot of perpendicular P from the origin to the plane be (x_1, y_1, z_1).

$2x + 3y + 4z - 12 = 0$

$\Rightarrow 2x + 3y + 4z = 12$ … (1)

The direction ratios of normal are 2, 3, and 4.

$\therefore \sqrt{(2)^2+(3)^2+(4)^2} = \sqrt{29}$

Dividing both sides of equation (1) by $\sqrt{29}$, we obtain

$$\frac{2}{\sqrt{29}}x+\frac{3}{\sqrt{29}}y+\frac{4}{\sqrt{29}}z=\frac{12}{\sqrt{29}}$$

This equation is of the form $lx + my + nz = d$, where l, m, n are the direction cosines of normal to the plane and d is the distance of normal from the origin.

The coordinates of the foot of the perpendicular are given by

(ld, md, nd).

Therefore, the coordinates of the foot of the perpendicular are

$$\left(\frac{2}{\sqrt{29}}.\frac{12}{\sqrt{29}},\frac{3}{\sqrt{29}}.\frac{12}{\sqrt{29}},\frac{4}{\sqrt{29}}.\frac{12}{\sqrt{29}}\right)\text{ i.e., }\left(\frac{24}{29},\frac{36}{49},\frac{48}{29}\right).$$

(b) Let the coordinates of the foot of perpendicular P from the origin to the plane be (x_1, y_1, z_1).

$3y+4z-6=0$

$\Rightarrow 0x+3y+4z=6$… (1)

The direction ratios of the normal are 0, 3, and 4.

$\therefore \sqrt{0+3^2+4^2}=5$

Dividing both sides of equation (1) by 5, we obtain

$$0x+\frac{3}{5}y+\frac{4}{5}z=\frac{6}{5}$$

This equation is of the form $lx + my + nz = d$, where l, m, n are the direction cosines of normal to the plane and d is the distance of normal from the origin.

The coordinates of the foot of the perpendicular are given by

(ld, md, nd).

Therefore, the coordinates of the foot of the perpendicular are

$$\left(0, \frac{3}{5}.\frac{6}{5}, \frac{4}{5}.\frac{6}{5}\right) \text{ i.e., } \left(0, \frac{18}{25}, \frac{24}{25}\right).$$

(c) Let the coordinates of the foot of perpendicular P from the origin to the plane be (x_1, y_1, z_1).

$x + y + z = 1$ … (1)

The direction ratios of the normal are 1, 1, and 1.

$$\therefore \sqrt{1^2+1^2+1^2} = \sqrt{3}$$

Dividing both sides of equation (1) by $\sqrt{3}$, we obtain

$$\frac{1}{\sqrt{3}}x + \frac{1}{\sqrt{3}}y + \frac{1}{\sqrt{3}}z = \frac{1}{\sqrt{3}}$$

This equation is of the form $lx + my + nz = d$, where l, m, n are the direction cosines of normal to the plane and d is the distance of normal from the origin.

The coordinates of the foot of the perpendicular are given by

(ld, md, nd).

Therefore, the coordinates of the foot of the perpendicular are

$$\left(\frac{1}{\sqrt{3}}.\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}.\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}.\frac{1}{\sqrt{3}}\right) \text{ i.e., } \left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right).$$

(d) Let the coordinates of the foot of perpendicular P from the origin to the plane be (x_1, y_1, z_1).

$5y + 8 = 0$

$\Rightarrow 0x - 5y + 0z = 8$ … (1)

The direction ratios of the normal are 0, −5, and 0.

$\therefore \sqrt{0+(-5)^2+0}=5$

Dividing both sides of equation (1) by 5, we obtain

$-y=\frac{8}{5}$

This equation is of the form *lx* + *my* + *nz* = *d*, where *l*, *m*, *n* are the direction cosines of normal to the plane and *d* is the distance of normal from the origin.

The coordinates of the foot of the perpendicular are given by

(*ld*, *md*, *nd*).

Therefore, the coordinates of the foot of the perpendicular are

$\left(0, -1\left(\frac{8}{5}\right), 0\right)$ i.e., $\left(0, -\frac{8}{5}, 0\right)$.

Question 5:

Find the vector and Cartesian equation of the planes

(a) that passes through the point (1, 0, −2) and the normal to the plane is $\hat{i}+\hat{j}-\hat{k}$.

(b) that passes through the point (1, 4, 6) and the normal vector to the plane is $\hat{i}-2\hat{j}+\hat{k}$.

Answer

(a) The position vector of point (1, 0, −2) is $\vec{a}=\hat{i}-2\hat{k}$

The normal vector $\vec{N}$ perpendicular to the plane is $\vec{N}=\hat{i}+\hat{j}-\hat{k}$

The vector equation of the plane is given by, $(\vec{r}-\vec{a}).\vec{N}=0$

$\Rightarrow \left[\vec{r}-\left(\hat{i}-2\hat{k}\right)\right].\left(\hat{i}+\hat{j}-\hat{k}\right)=0 \qquad ...(1)$

$\vec{r}$ is the position vector of any point P (*x*, *y*, *z*) in the plane.

$\therefore \vec{r}=x\hat{i}+y\hat{j}+z\hat{k}$

Therefore, equation (1) becomes

$$\left[\left(x\hat{i}+y\hat{j}+z\hat{k}\right)-\left(\hat{i}-2\hat{k}\right)\right].\left(\hat{i}+\hat{j}-\hat{k}\right)=0$$

$$\Rightarrow\left[(x-1)\hat{i}+y\hat{j}+(z+2)\hat{k}\right].\left(\hat{i}+\hat{j}-\hat{k}\right)=0$$

$$\Rightarrow(x-1)+y-(z+2)=0$$

$$\Rightarrow x+y-z-3=0$$

$$\Rightarrow x+y-z=3$$

This is the Cartesian equation of the required plane.

(b) The position vector of the point (1, 4, 6) is $\vec{a}=\hat{i}+4\hat{j}+6\hat{k}$

The normal vector $\vec{N}$ perpendicular to the plane is $\vec{N}=\hat{i}-2\hat{j}+\hat{k}$

The vector equation of the plane is given by, $(\vec{r}-\vec{a}).\vec{N}=0$

$$\Rightarrow\left[\vec{r}-\left(\hat{i}+4\hat{j}+6\hat{k}\right)\right].\left(\hat{i}-2\hat{j}+\hat{k}\right)=0 \qquad ...(1)$$

$\vec{r}$ is the position vector of any point P (*x*, *y*, *z*) in the plane.

$$\therefore\vec{r}=x\hat{i}+y\hat{j}+z\hat{k}$$

Therefore, equation (1) becomes

$$\left[\left(x\hat{i}+y\hat{j}+z\hat{k}\right)-\left(\hat{i}+4\hat{j}+6\hat{k}\right)\right].\left(\hat{i}-2\hat{j}+\hat{k}\right)=0$$

$$\Rightarrow\left[(x-1)\hat{i}+(y-4)\hat{j}+(z-6)\hat{k}\right]\cdot\left(\hat{i}-2\hat{j}+\hat{k}\right)=0$$

$$\Rightarrow(x-1)-2(y-4)+(z-6)=0$$

$$\Rightarrow x-2y+z+1=0$$

This is the Cartesian equation of the required plane.

Question 6:

Find the equations of the planes that passes through three points.

(a) (1, 1, −1), (6, 4, −5), (−4, −2, 3)

(b) (1, 1, 0), (1, 2, 1), (−2, 2, −1)

Answer

(a) The given points are A (1, 1, −1), B (6, 4, −5), and C (−4, −2, 3).

$$\begin{vmatrix} 1 & 1 & -1 \\ 6 & 4 & -5 \\ -4 & -2 & 3 \end{vmatrix} = (12-10)-(18-20)-(-12+16)$$
$$= 2+2-4$$
$$= 0$$

Since A, B, C are collinear points, there will be infinite number of planes passing through the given points.

(b) The given points are A (1, 1, 0), B (1, 2, 1), and C (−2, 2, −1).

$$\begin{vmatrix} 1 & 1 & 0 \\ 1 & 2 & 1 \\ -2 & 2 & -1 \end{vmatrix} = (-2-2)-(2+2) = -8 \neq 0$$

Therefore, a plane will pass through the points A, B, and C.

It is known that the equation of the plane through the points, (x_1, y_1, z_1), (x_2, y_2, z_2), and (x_3, y_3, z_3), is

$$\begin{vmatrix} x-x_1 & y-y_1 & z-z_1 \\ x_2-x_1 & y_2-y_1 & z_2-z_1 \\ x_3-x_1 & y_3-y_1 & z_3-z_1 \end{vmatrix} = 0$$

$$\Rightarrow \begin{vmatrix} x-1 & y-1 & z \\ 0 & 1 & 1 \\ -3 & 1 & -1 \end{vmatrix} = 0$$

$$\Rightarrow (-2)(x-1)-3(y-1)+3z = 0$$
$$\Rightarrow -2x-3y+3z+2+3 = 0$$
$$\Rightarrow -2x-3y+3z = -5$$
$$\Rightarrow 2x+3y-3z = 5$$

This is the Cartesian equation of the required plane.

Question 7:

Find the intercepts cut off by the plane $2x+y-z=5$

Answer

$$2x+y-z=5 \qquad \ldots(1)$$

Dividing both sides of equation (1) by 5, we obtain

$$\frac{2}{5}x+\frac{y}{5}-\frac{z}{5}=1$$

$$\Rightarrow \frac{x}{\frac{5}{2}}+\frac{y}{5}+\frac{z}{-5}=1 \qquad \ldots(2)$$

It is known that the equation of a plane in intercept form is $\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=1$, where *a*, *b*, *c* are the intercepts cut off by the plane at *x*, *y*, and *z* axes respectively.

Therefore, for the given equation,

$$a=\frac{5}{2}, b=5, \text{ and } c=-5$$

Thus, the intercepts cut off by the plane are $\frac{5}{2}$, 5, and -5.

Question 8:

Find the equation of the plane with intercept 3 on the *y*-axis and parallel to ZOX plane.

Answer

The equation of the plane ZOX is

y = 0

Any plane parallel to it is of the form, *y* = *a*

Since the *y*-intercept of the plane is 3,

∴ *a* = 3

Thus, the equation of the required plane is *y* = 3

Question 9:

Find the equation of the plane through the intersection of the planes

$3x-y+2z-4=0$ and $x+y+z-2=0$ and the point (2, 2, 1)

Answer

The equation of any plane through the intersection of the planes,

$3x - y + 2z - 4 = 0$ and $x + y + z - 2 = 0$, is

$$(3x-y+2z-4)+\alpha(x+y+z-2)=0, \text{ where } \alpha \in \mathrm{R} \qquad ...(1)$$

The plane passes through the point (2, 2, 1). Therefore, this point will satisfy equation (1).

$$\therefore (3\times 2-2+2\times 1-4)+\alpha(2+2+1-2)=0$$

$$\Rightarrow 2+3\alpha=0$$

$$\Rightarrow \alpha=-\frac{2}{3}$$

Substituting $\alpha=-\frac{2}{3}$ in equation (1), we obtain

$$(3x-y+2z-4)-\frac{2}{3}(x+y+z-2)=0$$

$$\Rightarrow 3(3x-y+2z-4)-2(x+y+z-2)=0$$

$$\Rightarrow (9x-3y+6z-12)-2(x+y+z-2)=0$$

$$\Rightarrow 7x-5y+4z-8=0$$

This is the required equation of the plane.

Question 10:

Find the vector equation of the plane passing through the intersection of the planes

$\vec{r}\cdot\left(2\hat{i}+2\hat{j}-3\hat{k}\right)=7,\ \vec{r}.\left(2\hat{i}+5\hat{j}+3\hat{k}\right)=9$ and through the point (2, 1, 3)

Answer

The equations of the planes are $\vec{r}\cdot\left(2\hat{i}+2\hat{j}-3\hat{k}\right)=7$ and $\vec{r}\cdot\left(2\hat{i}+5\hat{j}+3\hat{k}\right)=9$

$$\Rightarrow \vec{r}\cdot\left(2\hat{i}+2\hat{j}-3\hat{k}\right)-7=0 \qquad ...(1)$$

$$\vec{r}\cdot\left(2\hat{i}+5\hat{j}+3\hat{k}\right)-9=0 \qquad ...(2)$$

The equation of any plane through the intersection of the planes given in equations (1) and (2) is given by,

$\left[\vec{r}\cdot\left(2\hat{i}+2\hat{j}-3\hat{k}\right)-7\right]+\lambda\left[\vec{r}\cdot\left(2\hat{i}+5\hat{j}+3\hat{k}\right)-9\right]=0$, where $\lambda\in R$

$\vec{r}\cdot\left[\left(2\hat{i}+2\hat{j}-3\hat{k}\right)+\lambda\left(2\hat{i}+5\hat{j}+3\hat{k}\right)\right]=9\lambda+7$

$\vec{r}\cdot\left[\left(2+2\lambda\right)\hat{i}+\left(2+5\lambda\right)\hat{j}+\left(3\lambda-3\right)\hat{k}\right]=9\lambda+7 \qquad ...(3)$

The plane passes through the point (2, 1, 3). Therefore, its position vector is given by,

$\vec{r}=2\hat{i}+2\hat{j}+3\hat{k}$

Substituting in equation (3), we obtain

$\left(2\hat{i}+\hat{j}-3\hat{k}\right).\left[\left(2+2\lambda\right)\hat{i}+\left(2+5\lambda\right)\hat{j}+\left(3\lambda-3\right)\hat{k}\right]=9\lambda+7$

$\Rightarrow\left(2+2\lambda\right)+\left(2+5\lambda\right)+\left(3\lambda-3\right)=9\lambda+7$

$\Rightarrow 18\lambda-3=9\lambda+7$

$\Rightarrow 9\lambda=10$

$\Rightarrow \lambda=\frac{10}{9}$

Substituting $\lambda=\frac{10}{9}$ in equation (3), we obtain

$\vec{r}\cdot\left(\frac{38}{9}\hat{i}+\frac{68}{9}\hat{j}+\frac{3}{9}\hat{k}\right)=17$

$\Rightarrow\vec{r}\cdot\left(38\hat{i}+68\hat{j}+3\hat{k}\right)=153$

This is the vector equation of the required plane.

Question 11:

Find the equation of the plane through the line of intersection of the planes

$x+y+z=1$ and $2x+3y+4z=5$ which is perpendicular to the plane $x-y+z=0$

Answer

The equation of the plane through the intersection of the planes, $x+y+z=1$ and

$2x+3y+4z=5$, is

$\left(x+y+z-1\right)+\lambda\left(2x+3y+4z-5\right)=0$

$\Rightarrow\left(2\lambda+1\right)x+\left(3\lambda+1\right)y+\left(4\lambda+1\right)z-\left(5\lambda+1\right)=0 \qquad ...(1)$

The direction ratios, a_1, b_1, c_1, of this plane are (2λ + 1), (3λ + 1), and (4λ + 1).

The plane in equation (1) is perpendicular to $x-y+z=0$

Its direction ratios, a_2, b_2, c_2, are 1, −1, and 1.

Since the planes are perpendicular,

$$a_1a_2 + b_1b_2 + c_1c_2 = 0$$
$$\Rightarrow (2\lambda+1) - (3\lambda+1) + (4\lambda+1) = 0$$
$$\Rightarrow 3\lambda + 1 = 0$$
$$\Rightarrow \lambda = -\frac{1}{3}$$

Substituting $\lambda = -\frac{1}{3}$ in equation (1), we obtain

$$\frac{1}{3}x - \frac{1}{3}z + \frac{2}{3} = 0$$
$$\Rightarrow x - z + 2 = 0$$

This is the required equation of the plane.

Question 12:

Find the angle between the planes whose vector equations are

$\vec{r}\cdot\left(2\hat{i}+2\hat{j}-3\hat{k}\right)=5$ and $\vec{r}\cdot\left(3\hat{i}-3\hat{j}+5\hat{k}\right)=3$.

Answer

The equations of the given planes are $\vec{r}\cdot\left(2\hat{i}+2\hat{j}-3\hat{k}\right)=5$ and $\vec{r}\cdot\left(3\hat{i}-3\hat{j}+5\hat{k}\right)=3$

It is known that if $\vec{n}_1$ and $\vec{n}_2$ are normal to the planes, $\vec{r}\cdot\vec{n}_1 = d_1$ and $\vec{r}\cdot\vec{n}_2 = d_2$, then the angle between them, Q, is given by,

$$\cos Q = \left|\frac{\vec{n}_1\cdot\vec{n}_2}{|\vec{n}_1||\vec{n}_2|}\right| \qquad ...(1)$$

Here, $\vec{n}_1 = 2\hat{i}+2\hat{j}-3\hat{k}$ and $\vec{n}_2 = 3\hat{i}-3\hat{j}+5\hat{k}$

$$\therefore \vec{n}_1\cdot\vec{n}_2 = \left(2\hat{i}+2\hat{j}-3\hat{k}\right)\left(3\hat{i}-3\hat{j}+5\hat{k}\right) = 2.3 + 2.(-3) + (-3).5 = -15$$
$$|\vec{n}_1| = \sqrt{(2)^2+(2)^2+(-3)^2} = \sqrt{17}$$
$$|\vec{n}_2| = \sqrt{(3)^2+(-3)^2+(5)^2} = \sqrt{43}$$

Substituting the value of $\vec{n}\cdot\vec{n}_2$, $|\vec{n}_1|$ and $|\vec{n}_2|$ in equation (1), we obtain

$$\cos Q=\left|\frac{-15}{\sqrt{17}\cdot\sqrt{43}}\right|$$

$$\Rightarrow \cos Q=\frac{15}{\sqrt{731}}$$

$$\Rightarrow \cos Q^{-1}=\left(\frac{15}{\sqrt{731}}\right)$$

Question 13:

In the following cases, determine whether the given planes are parallel or perpendicular, and in case they are neither, find the angles between them.

(a) $7x+5y+6z+30=0$ and $3x-y-10z+4=0$

(b) $2x+y+3z-2=0$ and $x-2y+5=0$

(c) $2x-2y+4z+5=0$ and $3x-3y+6z-1=0$

(d) $2x-y+3z-1=0$ and $2x-y+3z+3=0$

(e) $4x+8y+z-8=0$ and $y+z-4=0$

Answer

The direction ratios of normal to the plane, $L_1: a_1x+b_1y+c_1z=0$, are a_1, b_1, c_1 and $L_2: a_1x+b_2y+c_2z=0$ are a_2, b_2, c_2.

$$L_1 \parallel L_2, \text{ if } \frac{a_1}{a_2}=\frac{b_1}{b_2}=\frac{c_1}{c_2}$$

$$L_1 \perp L_2, \text{ if } a_1a_2+b_1b_2+c_1c_2=0$$

The angle between L_1 and L_2 is given by,

$$Q=\cos^{-1}\left|\frac{a_1a_2+b_1b_2+c_1c_2}{\sqrt{a_1^2+b_1^2+c_1^2}\cdot\sqrt{a_2^2+b_2^2+c_2^2}}\right|$$

(a) The equations of the planes are $7x + 5y + 6z + 30 = 0$ and $3x - y - 10z + 4 = 0$

Here, $a_1 = 7$, $b_1 = 5$, $c_1 = 6$

$a_2 = 3,\ b_2 = -1, c_2 = -10$

$a_1a_2 + b_1b_2 + c_1c_2 = 7\times3 + 5\times(-1) + 6\times(-10) = -44 \neq 0$

Therefore, the given planes are not perpendicular.

$\frac{a_1}{a_2} = \frac{7}{3}, \frac{b_1}{b_2} = \frac{5}{-1} = -5, \frac{c_1}{c_2} = \frac{6}{-10} = \frac{-3}{5}$

It can be seen that, $\frac{a_1}{a_2} \neq \frac{b_1}{b_2} \neq, \frac{c_1}{c_2}$

Therefore, the given planes are not parallel.

The angle between them is given by,

$$Q = \cos^{-1}\left|\frac{7\times3 + 5\times(-1) + 6\times(-10)}{\sqrt{(7)^2 + (5)^2 + (6)^2} \times \sqrt{(3)^2 + (-1)^2 + (-10)^2}}\right|$$

$$= \cos^{-1}\left|\frac{21-5-60}{\sqrt{110}\times\sqrt{110}}\right|$$

$$= \cos^{-1}\frac{44}{110}$$

$$= \cos^{-1}\frac{2}{5}$$

(b) The equations of the planes are $2x + y + 3z - 2 = 0$ and $x - 2y + 5 = 0$

Here, $a_1 = 2,\ b_1 = 1,\ c_1 = 3$ and $a_2 = 1,\ b_2 = -2, c_2 = 0$

$\therefore a_1a_2 + b_1b_2 + c_1c_2 = 2\times1 + 1\times(-2) + 3\times0 = 0$

Thus, the given planes are perpendicular to each other.

(c) The equations of the given planes are $2x - 2y + 4z + 5 = 0$ and $3x - 3y + 6z - 1 = 0$

Here, $a_1 = 2,\ b_1 - 2,\ c_1 = 4$ and

$a_2 = 3,\ b_2 = -3,\ c_2 = 6 \quad a_1a_2 + b_1b_2 + c_1c_2 = 2\times3 + (-2)(-3) + 4\times6 = 6 + 6 + 24 = 36 \neq 0$

Thus, the given planes are not perpendicular to each other.

$\frac{a_1}{a_2} = \frac{2}{3}, \frac{b_1}{b_2} = \frac{-2}{-3} = \frac{2}{3} \text{and} \frac{c_1}{c_2} = \frac{4}{6} = \frac{2}{3}$

$\therefore \frac{a_1}{a_2} = \frac{b_1}{b_2} = \frac{c_1}{c_2}$

Thus, the given planes are parallel to each other.

(d) The equations of the planes are $2x - y + 3z - 1 = 0$ and $2x - y + 3z + 3 = 0$

Here, $a_1 = 2,\ b_1 = -1, c_1 = 3$ and $a_2 = 2,\ b_2 = -1,\ c_2 = 3$

$\frac{a_1}{a_2} = \frac{2}{2} = 1, \frac{b_1}{b_2} = \frac{-1}{-1} = 1 \text{and } \frac{c_1}{c_2} = \frac{3}{3} = 1$

$\therefore \frac{a_1}{a_2} = \frac{b_1}{b_2} = \frac{c_1}{c_2}$

Thus, the given lines are parallel to each other.

(e) The equations of the given planes are $4x + 8y + z - 8 = 0$ and $y + z - 4 = 0$

Here, $a_1 = 4,\ b_1 = 8,\ c_1 = 1$ and $a_2 = 0,\ b_2 = 1,\ c_2 = 1$

$a_1a_2 + b_1b_2 + c_1c_2 = 4 \times 0 + 8 \times 1 + 1 = 9 \neq 0$

Therefore, the given lines are not perpendicular to each other.

$\frac{a_1}{a_2} = \frac{4}{0},\ \frac{b_1}{b_2} = \frac{8}{1} = 8,\ \frac{c_1}{c_2} = \frac{1}{1} = 1$

$\therefore \frac{a_1}{a_2} \neq \frac{b_1}{b_2} \neq, \frac{c_1}{c_2}$

Therefore, the given lines are not parallel to each other.

The angle between the planes is given by,

$$Q = \cos^{-1}\left|\frac{4 \times 0 + 8 \times 1 + 1 \times 1}{\sqrt{4^2 + 8^2 + 1^2} \times \sqrt{0^2 + 1^2 + 1^2}}\right| = \cos^{-1}\left|\frac{9}{9 \times \sqrt{2}}\right| = \cos^{-1}\left(\frac{1}{\sqrt{2}}\right) = 45°$$

Question 14:

In the following cases, find the distance of each of the given points from the corresponding given plane.

Point Plane

(a) (0, 0, 0) $3x-4y+12z=3$

(b) (3, −2, 1) $2x-y+2z+3=0$

(c) (2, 3, −5) $x+2y-2z=9$

(d) (−6, 0, 0) $2x-3y+6z-2=0$

Answer

It is known that the distance between a point, $p(x_1, y_1, z_1)$, and a plane, $Ax + By + Cz = D$, is given by,

$$d=\left|\frac{Ax_1+By_1+Cz_1-D}{\sqrt{A^2+B^2+C^2}}\right| \qquad ...(1)$$

(a) The given point is (0, 0, 0) and the plane is $3x-4y+12z=3$

$$\therefore d=\left|\frac{3\times0-4\times0+12\times0-3}{\sqrt{(3)^2+(-4)^2+(12)^2}}\right|=\frac{3}{\sqrt{169}}=\frac{3}{13}$$

(b) The given point is (3, − 2, 1) and the plane is $2x-y+2z+3=0$

$$\therefore d=\left|\frac{2\times3-(-2)+2\times1+3}{\sqrt{(2)^2+(-1)^2+(2)^2}}\right|=\left|\frac{13}{3}\right|=\frac{13}{3}$$

(c) The given point is (2, 3, −5) and the plane is $x+2y-2z=9$

$$\therefore d=\left|\frac{2+2\times3-2(-5)-9}{\sqrt{(1)^2+(2)^2+(-2)^2}}\right|=\frac{9}{3}=3$$

(d) The given point is (−6, 0, 0) and the plane is $2x-3y+6z-2=0$

$$d=\left|\frac{2(-6)-3\times 0+6\times 0-2}{\sqrt{(2)^2+(-3)^2+(6)^2}}\right|=\left|\frac{-14}{\sqrt{49}}\right|=\frac{14}{7}=2$$

Miscellaneous Solutions

Question 1:

Show that the line joining the origin to the point (2, 1, 1) is perpendicular to the line determined by the points (3, 5, −1), (4, 3, −1).

Answer

Let OA be the line joining the origin, O (0, 0, 0), and the point, A (2, 1, 1).

Also, let BC be the line joining the points, B (3, 5, −1) and C (4, 3, −1).

The direction ratios of OA are 2, 1, and 1 and of BC are (4 − 3) = 1, (3 − 5) = −2, and (−1 + 1) = 0

OA is perpendicular to BC, if $a_1a_2 + b_1b_2 + c_1c_2 = 0$

$\therefore a_1a_2 + b_1b_2 + c_1c_2 = 2 \times 1 + 1(-2) + 1 \times 0 = 2 - 2 = 0$

Thus, OA is perpendicular to BC.

Question 2:

If l_1, m_1, n_1 and l_2, m_2, n_2 are the direction cosines of two mutually perpendicular lines, show that the direction cosines of the line perpendicular to both of these are $m_1n_2 - m_2n_1$, $n_1l_2 - n_2l_1$, $l_1m_2 - l_2m_1$.

Answer

It is given that l_1, m_1, n_1 and l_2, m_2, n_2 are the direction cosines of two mutually perpendicular lines. Therefore,

$$l_1l_2 + m_1m_2 + n_1n_2 = 0 \qquad ...(1)$$

$$l_1^2 + m_1^2 + n_1^2 = 1 \qquad ...(2)$$

$$l_2^2 + m_2^2 + n_2^2 = 1 \qquad ...(3)$$

Let l, m, n be the direction cosines of the line which is perpendicular to the line with direction cosines l_1, m_1, n_1 and l_2, m_2, n_2.

$$\therefore ll_1 + mm_1 + nn_1 = 0$$
$$ll_2 + mm_2 + nn_2 = 0$$
$$\therefore \frac{l}{m_1n_2 - m_2n_1} = \frac{m}{n_1l_2 - n_2l_1} = \frac{n}{l_1m_2 - l_2m_l}$$
$$\Rightarrow \frac{l^2}{(m_1n_2 - m_2n_1)^2} = \frac{m^2}{(n_1l_2 - n_2l_1)^2} = \frac{n^2}{(l_1m_2 - l_2m_l)^2}$$
$$\Rightarrow \frac{l^2}{(m_1n_2 - m_2n_1)^2} = \frac{m^2}{(n_1l_2 - n_2l_1)^2} = \frac{n^2}{(l_1m_2 - l_2m_2)^2}$$
$$= \frac{l^2 + m^2 + n^2}{(m_1n_2 - m_2n_1)^2 + (n_1l_2 - n_2l_1)^2 + (l_1m_2 - l_2m_l)^2} \quad ...(4)$$

l, *m*, *n* are the direction cosines of the line.

$\therefore l^2 + m^2 + n^2 = 1$ … (5)

It is known that,

$$\left(l_1^2 + m_1^2 + n_1^2\right)\left(l_2^2 + m_2^2 + n_2^2\right) - \left(l_1l_2 + m_1m_2 + n_1n_2\right)^2$$
$$= (m_1n_2 - m_2n_1)^2 + (n_1l_2 - n_2l_1)^2 + (l_1m_2 - l_2m_1)^2$$

From (1), (2), and (3), we obtain

$$\Rightarrow 1.1 - 0 = (m_1n_2 + m_2n_1)^2 + (n_1l_2 - n_2l_1)^2 + (l_1m_2 - l_2m_1)^2$$

$$\therefore (m_1n_2 - m_2n_1)^2 + (n_1l_2 - n_2l_1)^2 + (l_1m_2 - l_2m_1)^2 = 1 \quad ...(6)$$

Substituting the values from equations (5) and (6) in equation (4), we obtain

$$\frac{l^2}{(m_1n_2 - m_2n_1)^2} = \frac{m^2}{(n_2l_2 - n_2l_1)^2} = \frac{n^2}{(l_1m_2 - l_2m_1)^2} = 1$$
$$\Rightarrow l = m_1n_2 - m_2n_1, m = n_1l_2 - n_2l_1, n = l_1m_2 - l_2m_1$$

Thus, the direction cosines of the required line are $m_1n_2 - m_2n_1$, $n_1l_2 - n_2l_1$, and $l_1m_2 - l_2m_1$.

Question 3:

Find the angle between the lines whose direction ratios are *a*, *b*, *c* and *b* − *c*, *c* − *a*, *a* − *b*.

Answer

The angle *Q* between the lines with direction cosines, *a*, *b*, *c* and *b* − *c*, *c* − *a*, *a* − *b*, is given by,

$$\cos Q=\left|\frac{a(b-c)+b(c-a)+c(a-b)}{\sqrt{a^2+b^2+c^2}+\sqrt{(b-c)^2+(c-a)^2+(a-b)^2}}\right|$$

$$\Rightarrow \cos Q=0$$

$$\Rightarrow Q=\cos^{-1}0$$

$$\Rightarrow Q=90^\circ$$

Thus, the angle between the lines is 90°.

Question 4:

Find the equation of a line parallel to *x*-axis and passing through the origin.

Answer

The line parallel to *x*-axis and passing through the origin is *x*-axis itself.

Let A be a point on *x*-axis. Therefore, the coordinates of A are given by (*a*, 0, 0), where

a ∈ R.

Direction ratios of OA are (*a* − 0) = *a*, 0, 0

The equation of OA is given by,

$$\frac{x-0}{a}=\frac{y-0}{0}=\frac{z-0}{0}$$

$$\Rightarrow \frac{x}{1}=\frac{y}{0}=\frac{z}{0}=a$$

Thus, the equation of line parallel to *x*-axis and passing through origin is

$$\frac{x}{1}=\frac{y}{0}=\frac{z}{0}$$

Question 5:

If the coordinates of the points A, B, C, D be (1, 2, 3), (4, 5, 7), (−4, 3, −6) and (2, 9, 2) respectively, then find the angle between the lines AB and CD.

Answer

The coordinates of A, B, C, and D are (1, 2, 3), (4, 5, 7), (−4, 3, −6), and (2, 9, 2) respectively.

The direction ratios of AB are (4 − 1) = 3, (5 − 2) = 3, and (7 − 3) = 4

The direction ratios of CD are (2 −(− 4)) = 6, (9 − 3) = 6, and (2 −(−6)) = 8

It can be seen that, $\frac{a_1}{a_2}=\frac{b_1}{b_2}=\frac{c_1}{c_2}=\frac{1}{2}$

Therefore, AB is parallel to CD.

Thus, the angle between AB and CD is either 0° or 180°.

sQuestion 6:

If the lines $\frac{x-1}{-3}=\frac{y-2}{2k}=\frac{z-3}{2}$ and $\frac{x-1}{3k}=\frac{y-1}{1}=\frac{z-6}{-5}$ are perpendicular, find the value of *k*.

Answer

The direction of ratios of the lines, $\frac{x-1}{-3}=\frac{y-2}{2k}=\frac{z-3}{2}$ and $\frac{x-1}{3k}=\frac{y-1}{1}=\frac{z-6}{-5}$, are −3, 2*k*, 2 and 3*k*, 1, −5 respectively.

It is known that two lines with direction ratios, a_1, b_1, c_1 and a_2, b_2, c_2, are perpendicular, if $a_1a_2 + b_1b_2 + c_1c_2 = 0$

$$\therefore -3(3k)+2k\times 1+2(-5)=0$$

$$\Rightarrow -9k+2k-10=0$$

$$\Rightarrow 7k=-10$$

$$\Rightarrow k=\frac{-10}{7}$$

Therefore, for $k=-\frac{10}{7}$, the given lines are perpendicular to each other.

Question 7:

Find the vector equation of the plane passing through (1, 2, 3) and perpendicular to the plane $\vec{r}\cdot\left(\hat{i}+2\hat{j}-5\hat{k}\right)+9=0$

Answer

The position vector of the point (1, 2, 3) is $\vec{r_1}=\hat{i}+2\hat{j}+3\hat{k}$

The direction ratios of the normal to the plane, $\vec{r}\cdot\left(\hat{i}+2\hat{j}-5\hat{k}\right)+9=0$, are 1, 2, and −5 and the normal vector is $\vec{N}=\hat{i}+2\hat{j}-5\hat{k}$

The equation of a line passing through a point and perpendicular to the given plane is given by, $\vec{l}=\vec{r}+\lambda\vec{N},\ \lambda\in R$

$$\Rightarrow \vec{l}=\left(\hat{i}+2\hat{j}+3\hat{k}\right)+\lambda\left(\hat{i}+2\hat{j}-5\hat{k}\right)$$

Question 8:

Find the equation of the plane passing through (*a*, *b*, *c*) and parallel to the plane $\vec{r}\cdot\left(\hat{i}+\hat{j}+\hat{k}\right)=2$

Answer

Any plane parallel to the plane, $\vec{r_1}\cdot\left(\hat{i}+\hat{j}+\hat{k}\right)=2$, is of the form

$$\vec{r}\cdot\left(\hat{i}+\hat{j}+\hat{k}\right)=\lambda \qquad ...(1)$$

The plane passes through the point (*a*, *b*, *c*). Therefore, the position vector $\vec{r}$ of this point is $\vec{r}=a\hat{i}+b\hat{j}+c\hat{k}$

Therefore, equation (1) becomes

$$\left(a\hat{i}+b\hat{j}+c\hat{k}\right)\cdot\left(\hat{i}+\hat{j}+\hat{k}\right)=\lambda$$

$$\Rightarrow a+b+c=\lambda$$

Substituting $\lambda = a+b+c$ in equation (1), we obtain

$$\vec{r}\cdot\left(\hat{i}+\hat{j}+\hat{k}\right)=a+b+c \qquad ...(2)$$

This is the vector equation of the required plane.

Substituting $\vec{r} = x\hat{i}+y\hat{j}+z\hat{k}$ in equation (2), we obtain

$$\left(x\hat{i}+y\hat{j}+z\hat{k}\right)\cdot\left(\hat{i}+\hat{j}+\hat{k}\right)=a+b+c$$

$$\Rightarrow x+y+z=a+b+c$$

Question 9:

Find the shortest distance between lines $\vec{r}=6\hat{i}+2\hat{j}+2\hat{k}+\lambda\left(\hat{i}-2\hat{j}+2\hat{k}\right)$

and $\vec{r}=-4\hat{i}-\hat{k}+\mu\left(3\hat{i}-2\hat{j}-2\hat{k}\right)$.

Answer

The given lines are

$$\vec{r}=6\hat{i}+2\hat{j}+2\hat{k}+\lambda\left(\hat{i}-2\hat{j}+2\hat{k}\right) \qquad ...(1)$$

$$\vec{r}=-4\hat{i}-\hat{k}+\mu\left(3\hat{i}-2\hat{j}-2\hat{k}\right) \qquad ...(2)$$

It is known that the shortest distance between two lines, $\vec{r}=\vec{a}_1+\lambda\vec{b}_1$ and $\vec{r}=\vec{a}_2+\lambda\vec{b}_2$, is given by

$$d=\left|\frac{\left(\vec{b}_1\times\vec{b}_2\right)\cdot\left(\vec{a}_2-\vec{a}_1\right)}{\left|\vec{b}_1\times\vec{b}_2\right|}\right| \qquad ...(3)$$

Comparing $\vec{r}=\vec{a}_1+\lambda\vec{b}_1$ and $\vec{r}=\vec{a}_2+\lambda\vec{b}_2$ to equations (1) and (2), we obtain

$$\vec{a}_1=6\hat{i}+2\hat{j}+2\hat{k}$$

$$\vec{b}_1=\hat{i}-2\hat{j}+2\hat{k}$$

$$\vec{a}_2=-4\hat{i}-\hat{k}$$

$$\vec{b}_2=3\hat{i}-2\hat{j}-2\hat{k}$$

$$\Rightarrow \vec{a}_2-\vec{a}_1=\left(-4\hat{i}-\hat{k}\right)-\left(6\hat{i}+2\hat{j}+2\hat{k}\right)=-10\hat{i}-2\hat{j}-3\hat{k}$$

$$\Rightarrow \vec{b}_1 \times \vec{b}_2 = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & -2 & 2 \\ 3 & -2 & -2 \end{vmatrix} = (4+4)\hat{i} - (-2-6)\hat{j} + (-2+6)\hat{k} = 8\hat{i} + 8\hat{j} + 4\hat{k}$$

$$\therefore \left|\vec{b}_1 \times \vec{b}_2\right| = \sqrt{(8)^2 + (8)^2 + (4)^2} = 12$$

$$\left(\vec{b}_1 \times \vec{b}_2\right) \cdot \left(\vec{a}_2 - \vec{a}_1\right) = \left(8\hat{i} + 8\hat{j} + 4\hat{k}\right) \cdot \left(-10\hat{i} - 2\hat{j} - 3\hat{k}\right) = -80 - 16 - 12 = -108$$

Substituting all the values in equation (1), we obtain

$$d = \left|\frac{-108}{12}\right| = 9$$

Therefore, the shortest distance between the two given lines is 9 units.

Question 10:

Find the coordinates of the point where the line through (5, 1, 6) and (3, 4, 1) crosses the YZ-plane

Answer

It is known that the equation of the line passing through the points, (x_1, y_1, z_1) and (x_2, y_2, z_2), is $\frac{x - x_1}{x_2 - x_1} = \frac{y - y_1}{y_2 - y_1} = \frac{z - z_1}{z_2 - z_1}$

The line passing through the points, (5, 1, 6) and (3, 4, 1), is given by,

$$\frac{x-5}{3-5} = \frac{y-1}{4-1} = \frac{z-6}{1-6}$$

$$\Rightarrow \frac{x-5}{-2} = \frac{y-1}{3} = \frac{z-6}{-5} = k\,(\text{say})$$

$$\Rightarrow x = 5 - 2k,\ y = 3k + 1,\ z = 6 - 5k$$

Any point on the line is of the form $(5 - 2k, 3k + 1, 6 - 5k)$.

The equation of YZ-plane is $x = 0$

Since the line passes through YZ-plane,

$5 - 2k = 0$

$$\Rightarrow k = \frac{5}{2}$$

$$\Rightarrow 3k+1=3\times\frac{5}{2}+1=\frac{17}{2}$$

$$6-5k=6-5\times\frac{5}{2}=\frac{-13}{2}$$

Therefore, the required point is $\left(0,\frac{17}{2},\frac{-13}{2}\right)$.

Question 11:

Find the coordinates of the point where the line through (5, 1, 6) and (3, 4, 1) crosses the ZX − plane.

Answer

It is known that the equation of the line passing through the points, (x_1, y_1, z_1) and (x_2, y_2, z_2), is $\frac{x-x_1}{x_2-x_1}=\frac{y-y_1}{y_2-y_1}=\frac{z-z_1}{z_2-z_1}$

The line passing through the points, (5, 1, 6) and (3, 4, 1), is given by,

$$\frac{x-5}{3-5}=\frac{y-1}{4-1}=\frac{z-6}{1-6}$$

$$\Rightarrow \frac{x-5}{-2}=\frac{y-1}{3}=\frac{z-6}{-5}=k\,(\text{say})$$

$$\Rightarrow x=5-2k,\ y=3k+1,\ z=6-5k$$

Any point on the line is of the form $(5-2k, 3k+1, 6-5k)$.

Since the line passes through ZX-plane,

$$3k+1=0$$

$$\Rightarrow k=-\frac{1}{3}$$

$$\Rightarrow 5-2k=5-2\left(-\frac{1}{3}\right)=\frac{17}{3}$$

$$6-5\text{k}=6-5\left(-\frac{1}{3}\right)=\frac{23}{3}$$

Therefore, the required point is $\left(\frac{17}{3},0,\frac{23}{3}\right)$.

Question 12:

Find the coordinates of the point where the line through (3, −4, −5) and (2, − 3, 1) crosses the plane $2x + y + z = 7$).

Answer

It is known that the equation of the line through the points, (x_1, y_1, z_1) and (x_2, y_2, z_2), is

$$\frac{x-x_1}{x_2-x_1}=\frac{y-y_1}{y_2-y_1}=\frac{z-z_1}{z_2-z_1}$$

Since the line passes through the points, (3, −4, −5) and (2, −3, 1), its equation is given by,

$$\frac{x-3}{2-3}=\frac{y+4}{-3+4}=\frac{z+5}{1+5}$$

$$\Rightarrow \frac{x-3}{-1}=\frac{y+4}{1}=\frac{z+5}{6}=k\left(\text{say}\right)$$

$$\Rightarrow x=3-k,\ y=k-4,\ z=6k-5$$

Therefore, any point on the line is of the form $(3 - k, k - 4, 6k - 5)$.

This point lies on the plane, $2x + y + z = 7$

$\therefore 2(3 - k) + (k - 4) + (6k - 5) = 7$

$$\Rightarrow 5k-3=7$$

$$\Rightarrow k=2$$

Hence, the coordinates of the required point are $(3 - 2, 2 - 4, 6 \times 2 - 5)$ i.e., $(1, -2, 7)$.

Question 13:

Find the equation of the plane passing through the point (−1, 3, 2) and perpendicular to each of the planes $x + 2y + 3z = 5$ and $3x + 3y + z = 0$.

Answer

The equation of the plane passing through the point (−1, 3, 2) is

$a(x + 1) + b(y - 3) + c(z - 2) = 0$ … (1)

where, a, b, c are the direction ratios of normal to the plane.

It is known that two planes, $a_1x+b_1y+c_1z+d_1=0$ and $a_2x+b_2y+c_2z+d_2=0$, are perpendicular, if $a_1a_2+b_1b_2+c_1c_2=0$

Plane (1) is perpendicular to the plane, $x + 2y + 3z = 5$

$\therefore a\cdot 1+b\cdot 2+c\cdot 3=0$

$\Rightarrow a+2b+3c=0 \qquad ...(2)$

Also, plane (1) is perpendicular to the plane, $3x + 3y + z = 0$

$\therefore a\cdot 3+b\cdot 3+c\cdot 1=0$

$\Rightarrow 3a+3b+c=0 \qquad ...(3)$

From equations (2) and (3), we obtain

$$\frac{a}{2\times1-3\times3}=\frac{b}{3\times3-1\times1}=\frac{c}{1\times3-2\times3}$$

$$\Rightarrow \frac{a}{-7}=\frac{b}{8}=\frac{c}{-3}=k\left(\text{say}\right)$$

$$\Rightarrow a=-7k,\ b=8k,\ c=-3k$$

Substituting the values of a, b, and c in equation (1), we obtain

$$-7k\left(x+1\right)+8k\left(y-3\right)-3k\left(z-2\right)=0$$

$$\Rightarrow \left(-7x-7\right)+\left(8y-24\right)-3z+6=0$$

$$\Rightarrow -7x+8y-3z-25=0$$

$$\Rightarrow 7x-8y+3z+25=0$$

This is the required equation of the plane.

Question 14:

If the points (1, 1, p) and (−3, 0, 1) be equidistant from the plane

$\vec{r}\cdot\left(3\hat{i}+4\hat{j}-12\hat{k}\right)+13=0$, then find the value of p.

Answer

The position vector through the point (1, 1, p) is $\vec{a}_1=\hat{i}+\hat{j}+p\hat{k}$

Similarly, the position vector through the point (−3, 0, 1) is

$\vec{a}_2=-4\hat{i}+\hat{k}$

The equation of the given plane is $\vec{r}\cdot\left(3\hat{i}+4\hat{j}-12\hat{k}\right)+13=0$

It is known that the perpendicular distance between a point whose position vector is

$\vec{a}$ and the plane, $\vec{r}\cdot\vec{N}=d$, is given by, $D=\frac{\left|\vec{a}\cdot\vec{N}-d\right|}{\left|\vec{N}\right|}$

Here, $\vec{N}=3\hat{i}+4\hat{j}-12\hat{k}$ and $d=-13$

Therefore, the distance between the point (1, 1, p) and the given plane is

$$D_1=\frac{\left|\left(\hat{i}+\hat{j}+p\hat{k}\right)\cdot\left(3\hat{i}+4\hat{j}-12\hat{k}\right)+13\right|}{\left|3\hat{i}+4\hat{j}-12\hat{k}\right|}$$

$$\Rightarrow D_1=\frac{\left|3+4-12p+13\right|}{\sqrt{3^2+4^2+(-12)^2}}$$

$$\Rightarrow D_1=\frac{\left|20-12p\right|}{13} \qquad ...(1)$$

Similarly, the distance between the point (−3, 0, 1) and the given plane is

$$D_2=\frac{\left|\left(-3\hat{i}+\hat{k}\right)\cdot\left(3\hat{i}+4\hat{j}-12\hat{k}\right)+13\right|}{\left|3\hat{i}+4\hat{j}-12\hat{k}\right|}$$

$$\Rightarrow D_2=\frac{\left|-9-12+13\right|}{\sqrt{3^2+4^2+(-12)^2}}$$

$$\Rightarrow D_2=\frac{8}{13} \qquad ...(2)$$

It is given that the distance between the required plane and the points, (1, 1, p) and (−3, 0, 1), is equal.

$\therefore D_1 = D_2$

$$\Rightarrow \frac{\left|20-12p\right|}{13}=\frac{8}{13}$$

$\Rightarrow 20-12p=8$ or $-(20-12p)=8$

$\Rightarrow 12p=12$ or $12p=28$

$\Rightarrow p=1$ or $p=\frac{7}{3}$

Question 15:

Find the equation of the plane passing through the line of intersection of the planes

$\vec{r}\cdot(\hat{i}+\hat{j}+\hat{k})=1$ and $\vec{r}\cdot(2\hat{i}+3\hat{j}-\hat{k})+4=0$ and parallel to x-axis.

Answer

The given planes are

$$\vec{r}\cdot(\hat{i}+\hat{j}+\hat{k})=1$$

$$\Rightarrow \vec{r}\cdot(\hat{i}+\hat{j}+\hat{k})-1=0$$

$$\vec{r}\cdot(2\hat{i}+3\hat{j}-\hat{k})+4=0$$

The equation of any plane passing through the line of intersection of these planes is

$$\left[\vec{r}\cdot(\hat{i}+\hat{j}+\hat{k})-1\right]+\lambda\left[\vec{r}\cdot(2\hat{i}+3\hat{j}-\hat{k})+4\right]=0$$

$$\vec{r}\cdot\left[(2\lambda+1)\hat{i}+(3\lambda+1)\hat{j}+(1-\lambda)\hat{k}\right]+(4\lambda+1)=0 \qquad ...(1)$$

Its direction ratios are $(2\lambda + 1)$, $(3\lambda + 1)$, and $(1 - \lambda)$.

The required plane is parallel to x-axis. Therefore, its normal is perpendicular to x-axis.

The direction ratios of x-axis are 1, 0, and 0.

$$\therefore 1.(2\lambda+1)+0(3\lambda+1)+0(1-\lambda)=0$$

$$\Rightarrow 2\lambda+1=0$$

$$\Rightarrow \lambda=-\frac{1}{2}$$

Substituting $\lambda=-\frac{1}{2}$ in equation (1), we obtain

$$\Rightarrow \vec{r}\cdot\left[-\frac{1}{2}\hat{j}+\frac{3}{2}\hat{k}\right]+(-3)=0$$

$$\Rightarrow \vec{r}(\hat{j}-3\hat{k})+6=0$$

Therefore, its Cartesian equation is $y - 3z + 6 = 0$

This is the equation of the required plane.

Question 16:

If O be the origin and the coordinates of P be (1, 2, −3), then find the equation of the plane passing through P and perpendicular to OP.

Answer

The coordinates of the points, O and P, are (0, 0, 0) and (1, 2, −3) respectively.

Therefore, the direction ratios of OP are (1 − 0) = 1, (2 − 0) = 2, and (−3 − 0) = −3

It is known that the equation of the plane passing through the point (x_1, y_1 z_1) is

$a(x-x_1)+b(y-y_1)+c(z-z_1)=0$ where, a, b, and c are the direction ratios of normal.

Here, the direction ratios of normal are 1, 2, and −3 and the point P is (1, 2, −3).

Thus, the equation of the required plane is

$$1(x-1)+2(y-2)-3(z+3)=0$$
$$\Rightarrow x+2y-3z-14=0$$

Question 17:

Find the equation of the plane which contains the line of intersection of the planes

$\vec{r}\cdot(\hat{i}+2\hat{j}+3\hat{k})-4=0$, $\vec{r}\cdot(2\hat{i}+\hat{j}-\hat{k})+5=0$ and which is perpendicular to the plane

$\vec{r}\cdot(5\hat{i}+3\hat{j}-6\hat{k})+8=0$.

Answer

The equations of the given planes are

$$\vec{r}\cdot(\hat{i}+2\hat{j}+3\hat{k})-4=0 \qquad ...(1)$$
$$\vec{r}\cdot(2\hat{i}+\hat{j}-\hat{k})+5=0 \qquad ...(2)$$

The equation of the plane passing through the line intersection of the plane given in equation (1) and equation (2) is

$$\left[\vec{r}\cdot(\hat{i}+2\hat{j}+3\hat{k})-4\right]+\lambda\left[\vec{r}\cdot(2\hat{i}+\hat{j}-\hat{k})+5\right]=0$$
$$\vec{r}\cdot\left[(2\lambda+1)\hat{i}+(\lambda+2)\hat{j}+(3-\lambda)\hat{k}\right]+(5\lambda-4)=0 \qquad ...(3)$$

The plane in equation (3) is perpendicular to the plane, $\vec{r}\cdot\left(5\hat{i}+3\hat{j}-6\hat{k}\right)+8=0$

$\therefore 5(2\lambda+1)+3(\lambda+2)-6(3-\lambda)=0$

$\Rightarrow 19\lambda-7=0$

$\Rightarrow \lambda=\dfrac{7}{19}$

Substituting $\lambda=\dfrac{7}{19}$ in equation (3), we obtain

$\Rightarrow \vec{r}\cdot\left[\dfrac{33}{19}\hat{i}+\dfrac{45}{19}\hat{j}+\dfrac{50}{19}\hat{k}\right]\dfrac{-41}{19}=0$

$\Rightarrow \vec{r}\cdot\left(33\hat{i}+45\hat{j}+50\hat{k}\right)-41=0 \qquad ...(4)$

This is the vector equation of the required plane.

The Cartesian equation of this plane can be obtained by substituting $\vec{r}=x\hat{i}+y\hat{j}+z\hat{k}$ in equation (3).

$\left(x\hat{i}+y\hat{j}+z\hat{k}\right)\cdot\left(33\hat{i}+45\hat{j}+50\hat{k}\right)-41=0$

$\Rightarrow 33x+45y+50z-41=0$

Question 18:

Find the distance of the point (−1, −5, −10) from the point of intersection of the line $\vec{r}=2\hat{i}-\hat{j}+2\hat{k}+\lambda\left(3\hat{i}+4\hat{j}+2\hat{k}\right)$ and the plane $\vec{r}.\left(\hat{i}-\hat{j}+\hat{k}\right)=5$.

Answer

The equation of the given line is

$\vec{r}\cdot=2\hat{i}-\hat{j}+2\hat{k}+\lambda\left(3\hat{i}+4\hat{j}+2\hat{k}\right) \qquad ...(1)$

The equation of the given plane is

$\vec{r}.\left(\hat{i}-\hat{j}+\hat{k}\right)=5 \qquad ...(2)$

Substituting the value of $\vec{r}$ from equation (1) in equation (2), we obtain

$$\left[2\hat{i}-\hat{j}+2\hat{k}+\lambda\left(3\hat{i}+4\hat{j}+2\hat{k}\right)\right]\cdot\left(\hat{i}-\hat{j}+\hat{k}\right)=5$$

$$\Rightarrow\left[(3\lambda+2)\hat{i}+(4\lambda-1)\hat{j}+(2\lambda+2)\hat{k}\right]\cdot\left(\hat{i}-\hat{j}+\hat{k}\right)=5$$

$$\Rightarrow(3\lambda+2)-(4\lambda-1)+(2\lambda+2)=5$$

$$\Rightarrow\lambda=0$$

Substituting this value in equation (1), we obtain the equation of the line as

$$\vec{r}=2\hat{i}-\hat{j}+2\hat{k}$$

This means that the position vector of the point of intersection of the line and the plane is $\vec{r}=2\hat{i}-\hat{j}+2\hat{k}$

This shows that the point of intersection of the given line and plane is given by the coordinates, (2, −1, 2). The point is (−1, −5, −10).

The distance *d* between the points, (2, −1, 2) and (−1, −5, −10), is

$$d=\sqrt{(-1-2)^2+(-5+1)^2+(-10-2)^2}=\sqrt{9+16+144}=\sqrt{169}=13$$

Question 19:

Find the vector equation of the line passing through (1, 2, 3) and parallel to the planes $\vec{r}=\left(\hat{i}-\hat{j}+2\hat{k}\right)=5$ and $\vec{r}\cdot\left(3\hat{i}+\hat{j}+\hat{k}\right)=6$

Answer

Let the required line be parallel to vector $\vec{b}$ given by,

$$\vec{b}=b_1\hat{i}+b_2\hat{j}+b_3\hat{k}$$

The position vector of the point (1, 2, 3) is $\vec{a}=\hat{i}+2\hat{j}+3\hat{k}$

The equation of line passing through (1, 2, 3) and parallel to $\vec{b}$ is given by,

$$\vec{r}=\vec{a}+\lambda\vec{b}$$

$$\Rightarrow\vec{r}\left(\hat{i}+2\hat{j}+3\hat{k}\right)+\lambda\left(b_1\hat{i}+b_2\hat{j}+b_3\hat{k}\right)\qquad\ldots(1)$$

The equations of the given planes are

$$\vec{r}\cdot\left(\hat{i}-\hat{j}+2\hat{k}\right)=5 \qquad ...(2)$$

$$\vec{r}\cdot\left(3\hat{i}+\hat{j}+\hat{k}\right)=6 \qquad ...(3)$$

The line in equation (1) and plane in equation (2) are parallel. Therefore, the normal to the plane of equation (2) and the given line are perpendicular.

$$\Rightarrow\left(\hat{i}-\hat{j}+2\hat{k}\right)\cdot\lambda\left(b_1\hat{i}+b_2\hat{j}+b_3\hat{k}\right)=0$$

$$\Rightarrow\lambda\left(b_1-b_2+2b_3\right)=0$$

$$\Rightarrow b_1-b_2+2b_3=0 \qquad ...(4)$$

Similarly, $\left(3\hat{i}+\hat{j}+\hat{k}\right)\cdot\lambda\left(b_1\hat{i}+b_2\hat{j}+b_3\hat{k}\right)=0$

$$\Rightarrow\lambda\left(3b_1+b_2+b_3\right)=0$$

$$\Rightarrow 3b_1+b_2+b_3=0 \qquad ...(5)$$

From equations (4) and (5), we obtain

$$\frac{b_1}{(-1)\times1-1\times2}=\frac{b_2}{2\times3-1\times1}=\frac{b_3}{1\times1-3(-1)}$$

$$\Rightarrow\frac{b_1}{-3}=\frac{b_2}{5}=\frac{b_3}{4}$$

Therefore, the direction ratios of $\vec{b}$ are −3, 5, and 4.

$$\therefore\vec{b}=b_1\hat{i}+b_2\hat{j}+b_3\hat{k}=-3\hat{i}+5\hat{j}+4\hat{k}$$

Substituting the value of $\vec{b}$ in equation (1), we obtain

$$\vec{r}=\left(\hat{i}+2\hat{j}+3\hat{k}\right)+\lambda\left(-3\hat{i}+5\hat{j}+4\hat{k}\right)$$

This is the equation of the required line.

Question 20:

Find the vector equation of the line passing through the point (1, 2, − 4) and

perpendicular to the two lines: $\frac{x-8}{3}=\frac{y+19}{-16}=\frac{z-10}{7}$ and $\frac{x-15}{3}=\frac{y-29}{8}=\frac{z-5}{-5}$

Answer

Let the required line be parallel to the vector $\vec{b}$ given by, $\vec{b}=b_1\hat{i}+b_2\hat{j}+b_3\hat{k}$

The position vector of the point (1, 2, − 4) is $\vec{a} = \hat{i} + 2\hat{j} - 4\hat{k}$

The equation of the line passing through (1, 2, −4) and parallel to vector $\vec{b}$ is

$\vec{r} = \vec{a} + \lambda \vec{b}$

$\Rightarrow \vec{r}\left(\hat{i} + 2\hat{j} - 4\hat{k}\right) + \lambda\left(b_1\hat{i} + b_2\hat{j} + b_3\hat{k}\right) \qquad ...(1)$

The equations of the lines are

$$\frac{x-8}{3} = \frac{y+19}{-16} = \frac{z-10}{7} \qquad ...(2)$$

$$\frac{x-15}{3} = \frac{y-29}{8} = \frac{z-5}{-5} \qquad ...(3)$$

Line (1) and line (2) are perpendicular to each other.

$\therefore 3b_1 - 16b_2 + 7b_3 = 0 \qquad ...(4)$

Also, line (1) and line (3) are perpendicular to each other.

$\therefore 3b_1 + 8b_2 - 5b_3 = 0 \qquad ...(5)$

From equations (4) and (5), we obtain

$$\frac{b_1}{(-16)(-5) - 8\times 7} = \frac{b_2}{7\times 3 - 3(-5)} = \frac{b_3}{3\times 8 - 3(-16)}$$

$$\Rightarrow \frac{b_1}{24} = \frac{b_2}{36} = \frac{b_3}{72}$$

$$\Rightarrow \frac{b_1}{2} = \frac{b_2}{3} = \frac{b_3}{6}$$

∴Direction ratios of $\vec{b}$ are 2, 3, and 6.

$\therefore \vec{b} = 2\hat{i} + 3\hat{j} + 6\hat{k}$

Substituting $\vec{b} = 2\hat{i} + 3\hat{j} + 6\hat{k}$ in equation (1), we obtain

$\vec{r} = \left(\hat{i} + 2\hat{j} - 4\hat{k}\right) + \lambda\left(2\hat{i} + 3\hat{j} + 6\hat{k}\right)$

This is the equation of the required line.

Question 21:

Prove that if a plane has the intercepts *a*, *b*, *c* and is at a distance of *P* units from the

origin, then $\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}=\frac{1}{p^2}$

Answer

The equation of a plane having intercepts *a*, *b*, *c* with *x*, *y*, and *z* axes respectively is given by,

$$\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=1 \qquad \ldots(1)$$

The distance (*p*) of the plane from the origin is given by,

$$p=\left|\frac{\frac{0}{a}+\frac{0}{b}+\frac{0}{c}-1}{\sqrt{\left(\frac{1}{a}\right)^2+\left(\frac{1}{b}\right)^2+\left(\frac{1}{c}\right)^2}}\right|$$

$$\Rightarrow p=\frac{1}{\sqrt{\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}}}$$

$$\Rightarrow p^2=\frac{1}{\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}}$$

$$\Rightarrow \frac{1}{p^2}=\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}$$

Question 22:

Distance between the two planes: $2x+3y+4z=4$ and $4x+6y+8z=12$ is

(A)2 units (B)4 units (C)8 units

(D) $\frac{2}{\sqrt{29}}$ units

Answer

The equations of the planes are

$$2x+3y+4z=4 \qquad \ldots(1)$$

$4x+6y+8z=12$

$\Rightarrow 2x+3y+4z=6 \qquad ...(2)$

It can be seen that the given planes are parallel.

It is known that the distance between two parallel planes, $ax + by + cz = d_1$ and $ax + by + cz = d_2$, is given by,

$$D=\left|\frac{d_2-d_1}{\sqrt{a^2+b^2+c^2}}\right|$$

$$\Rightarrow D=\left|\frac{6-4}{\sqrt{(2)^2+(3)^2+(4)^2}}\right|$$

$$D=\frac{2}{\sqrt{29}}$$

Thus, the distance between the lines is $\frac{2}{\sqrt{29}}$ units.

Hence, the correct answer is D.

Question 23:

The planes: $2x - y + 4z = 5$ and $5x - 2.5y + 10z = 6$ are

(A) Perpendicular (B) Parallel (C) intersect y-axis

(C) passes through $\left(0,0,\frac{5}{4}\right)$

Answer

The equations of the planes are

$2x - y + 4z = 5$ … (1)

$5x - 2.5y + 10z = 6$ … (2)

It can be seen that,

$$\frac{a_1}{a_2} = \frac{2}{5}$$

$$\frac{b_1}{b_2} = \frac{-1}{-2.5} = \frac{2}{5}$$

$$\frac{c_1}{c_2} = \frac{4}{10} = \frac{2}{5}$$

$$\therefore \frac{a_1}{a_2} = \frac{b_1}{b_2} = \frac{c_1}{c_2}$$

Therefore, the given planes are parallel.
Hence, the correct answer is B.