MATHEMATICS

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AIM POINT
MATHEMATICS
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DIFFERENTIAL EQUATIONS

& Their Properties

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THINGS TO REMEMBER

✷ Differential Equation

An equation involving independent variable, dependent variable, derivatives of dependent variable with respect to independent variable an constant, is called a differential equation.

eg, $x\frac{dy}{dx}+xy\frac{d^2y}{dx^2}+4=0, \frac{dy}{dx}+y\cos x=\sin x$ are differential equation.

A differential equation is said to be ordinary, if the differential coefficients have reference to a single indeendent variable only otherwise it is said to be partial (there are two or more independent variables.)

✷ Order and Degree of a Differential Equation

The order of hightest differential coefficient (or highest order derivative) appearing in a differential equation is the order of differential equation. eg, Differential equation $\frac{d^3y}{dx^3}+\frac{dy}{dx}+xy\frac{d^2y}{dx^2}+4=0$ is of order 3. The differential equation $\frac{d^2y}{dx^2}+\left(\frac{dy}{dx}\right)^3=0$ is of order 2.

The highest exponent of the highest derivative is called degree of a differential equation provided exponent of each derivative and the unknown variable appearing in the differential equation is a non-negative integer, eg,

1. $\frac{d^2y}{dx^2}+\sqrt{\frac{dy}{dx}}+x=0 \Rightarrow \frac{dy}{dx}=\left(\frac{d^2y}{dx^2}+x\right)^2$

$$\Rightarrow \quad \left(\frac{d^2y}{dx^2}\right)^2-\left(\frac{dy}{dx}\right)+2x\frac{d^2y}{dx^2}+x^2=0$$

Here, degree is 2.

2. $\left(\frac{d^3y}{dx^3}\right)^{2/3}+x+y=0 \Rightarrow \left(\frac{d^3y}{dx^3}\right)^2 \backslash =(-x-y)^3$

So, degree is 2.

✷ Formation of a Differential Equation

If an equation in independent, dependent variable involving some arbitrary constants is given, then a differential equation is obtained as follows

1. Differentiate the given equation w.r.t. the independent variable (say x) as many times as the number of arbitrary constants in it.
2. Eliminate the arbitrary constant.
3. The obtained equation is the required diffeerential equation ie, if we have an equation $f(x,y,c_1,c_2,....c_n)=0$ constaining n arbitrary constants c1, c2,.......,cn, then by differentiating this n time, we shall get n-equations.

Now, among these n-equation and the given equation, in all (n + 1) equations. If the n arbitraty constants $c_1, c_2, ..., c_n$, are eliminated, we shall evidently get a differential equation of the nth order. For there being n differentiation, the resulting equation must contain a derivative of the nth order.

✷ Solution of a Differential Equation

A solution or integral of a differential equation is a relation between the variables not involving the differential coefficients such that this relation and the derivatives obtained from it satisfy the given differential equation.

A solution of a differential equation which contans arbitrary constants as many as the order of the differential equation is called 'Genral solution'. Other solutions, obtained by giving particular values to the arbitrary constants in te general solution, are called 'particular solution'.

✷ Differential Equation of First Order and First Degree

The most general differential equation of first order and first degree is

$$\frac{dy}{dx} = f(x, y) \text{ or } \frac{f(x, y)}{g(x, y)}$$

All differential equations of first order cannot be always solved. They can be solved by the following methods.

1. Variables Separable Method

Equations in which the variables are separable are those equations which can be expressed that the coefficient of dx is only a function of x and that of dy is only a function of y.

Thus, the general form of such an equation is $f(x)dx + g(y)dy = 0$. The solution of this equation is obtained by integrating $f(x)$ and g(y) with respect to x and y repectively.

$$\int f(x)dx = \int g(y)dy + c$$

2. Differential Equations Reducible to variable Separable Method

Sometimes differential equation of the first order cannot be solved directly by variable separable method. But by come substitution, we can reduce it to a differential equation with separable varible. Let the differential equation is of the form.

$$\frac{dy}{dx} = f(ax + by + c)$$

then pur $ax + by + c = t$ and $\frac{dy}{dx} = \frac{dt}{dx} - a$ and then apply variable separable method.

3. Homigeneous Differential Equation

A differential equation of the form $\frac{dy}{dx} = \frac{f(x, y)}{g(x, y)}$ where $f(x, y)$ and $g(x, y)$ are homogeneous functions (ie, degree of each term in x and y is same) of x, y and of the same degree, is said to be homogeneous.

For solving such equations put y = vx and $\frac{dy}{dx} = v + x\frac{dv}{dx}$ so that the dependent variable y is changed to another variable v. Then apply variable separable method.

4. Differential Equation Reducible to Homogeneous Form

The equation of the form

$$\frac{dy}{dx} = \frac{a_1x + b_1x + c_1}{a_2x + b_2x + c_2} \qquad \left(\frac{a_1}{a_2} \neq \frac{b_1}{b_2}\right)$$

can be reduce to a homogeneous form by putting x = X + h and y = Y + k (h,k are constant.)

and $$\frac{dy}{dx} = \frac{dY}{dX}$$

Then equation reduces to

$$\frac{dY}{dX} = \frac{a_1X + b_1Y + c_1 + a_1h + b_1k}{a_2X + b_2Y + c_2 + a_2h + b_2k}$$

Put $a_1h + b_1k + c_1 = 0$ ...(i)

Put $a_2h + b_2k + c_2 = 0$ ...(ii)

On solving Eqs. (i) and (ii) for h and k we get the values of h and k.

Now, $$\frac{dY}{dX} = \frac{a_1X + b_1Y}{a_2X + b_2Y}$$

Which is a homoeneous differential equation. If $\frac{a_1}{a_2} = \frac{b_1}{b_2}$, then put $a_1x + b_1y = t$ and the it reduces to variable separable differential equation.

5. Linear Differential Equation

A differential equation is said to be linear if the dependent variable and its differential coeffi cients occur in the first degree only and are not multiplied together.

The most general form of a linear equation of the first order is

$$\frac{dy}{dx} + Py = Q$$

where P and Q are any function of x.

To solve such equations find entegrating factor.

ie, $$IF = e^{\int Pdx}$$

Then solution is givne by

$$y \cdot IF = \int Q \cdot IF dx + c$$

6. Differential Equation Reducible to Linear Form

An equation of the form

$$f'(y)\frac{dy}{dx} + P f(y) = Q$$

where P and Q are constant or function of x alone, can be reduced to linear form by putting.

$$f(y) = v \quad \Rightarrow \quad f'(y)\frac{dy}{dx} = \frac{dv}{dx}$$

Then Eq. (i) becomes $\frac{dv}{dx} + Pv = Q$

which is linear in v and x.

7. A Special Case Bernoulli's Equation

An equation of the form

where P and Q are function of x only and $n \neq 0, 1$ is known as bernoulli's differential equation.

It is easy to reduce the equation into linear form as below.

On dividing both sides by y^n, we get.

$$y^{-n} + Py^{1-n} = Q$$

Put $\quad y^{1-n} = z \quad \Rightarrow \quad (1-n)\, y^{-n}\frac{dy}{dx} = \frac{dz}{dx}$

$\therefore$ Given equation becomes + $(1-n)Pz = (1-n)Q$

Which is a linear differential equation is z.

Here, $\quad IF = e^{\int (1-n)Pdx}$

Required solution is

$$z(IF)y \cdot IF = \int \{(1-n)Q \cdot e^{\int (1-n)Pdx}\}dx$$

✷ <u>**Orthogonal Trajectory**</u>

A curve which cuts every member of a given family of curves at right angle, is called an orthogonal trajectory of the family. To find the orthogonal trajectory

1. Let $f(x, y, c) = 0$ ...(i)

   be the equation of given family of curves, where c is a parameter (or arbitrary constant).

2. Find the differential equation of Eq. (i) and then substitue $-\frac{dx}{dy}$ for $\frac{dy}{dx}$ in the differential equation.

   Thus we will get the differential equation of the orthogonal trajectories.

3. Now, solve this differential equation to obtain general solution which will give us the required orthogonal trajectories.

✷ Some Exact Differentiation

1. $d\left(\frac{y}{x}\right)=\frac{xdy-ydx}{x^2}$

2. $d\left(\frac{y}{x}\right)=\frac{xdy-ydx}{x^2}$

3. $d[\log(x\ y)]=\frac{xdy+ydx}{xy}$

4. $d\left[\tan^{-1}\frac{y}{x}\right]=\frac{xdy-ydx}{xy}$

5. $d\left[\log\left(\frac{y}{x}\right)\right]=\frac{xdy-ydx}{xy}$

Note :

- If linear equation is of the form $\frac{dy}{dx}$ + Px = Q where P and Q are the functions of y. Then IF = $e^{\int Pdy}$ and solution is given by

$$x \cdot \text{IF} = \int \text{Q} \cdot \text{IF}\ dy + c$$

- If equation is of the form , where n= 0, 1 and P and Q are function of y only. Then divide by xn and put $x^{1-n} = z \Rightarrow (1-n)x^{-n}\frac{dx}{dy} = \frac{dz}{dx}$.