MATHEMATICS

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MATHEMATICS
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XIth, XIIth, TARGET IIT-JEE (MAIN + ADVANCE) & COMPETITIVE EXAM. FOR XI (PQRS)

FUNCTIONS

& Their Properties

CONTENTS

THINGS TO REMEMBER

1. $[-n] = -[n]$
2. $[x + k] = [x] + k$ for any integer k.
3. $[-x] = -[x] - 1$
4. $[x] + [-x] = \begin{cases} -1, \text{if } x \notin Z \\ 0, \text{if } x \in Z \end{cases}$
5. $[x] - [-x] = \begin{cases} 2[x]+1 & \text{, if } x \notin Z \\ 2[x] & \text{, if } x \in Z \end{cases}$
6. $[x] \geq k \Rightarrow x \geq k$, where $k \in Z$
7. $[x] \leq k \Rightarrow x > k + 1$, where $k \in Z$
8. $[x] > k \Rightarrow x \geq k + 1$, where $k \in Z$
9. $[x] < k \Rightarrow x < k$, where $k \in Z$
10. $[x + y] = [x] + [y + x - [x]]$ for all $x, y \in R$
11. $[x] + \left[x+\frac{1}{n}\right] + \left[x+\frac{2}{n}\right] + \ldots + \left[x+\frac{n-1}{n}\right] = [nx]$, $n \in N$.
12. $\lceil -n \rceil = -\lceil n \rceil$
13. $\lceil -x \rceil = -\lceil x \rceil - 1$, where $x \in R - Z$
14. $\lceil -x + n \rceil = \lceil x \rceil + n$, where $x \in R - Z$ and $n \in Z$
15. $\lceil x \rceil + \lceil -x \rceil = \begin{cases} 1, \text{if } x \notin Z \\ 0, \text{if } x \in Z \end{cases}$
16. $\lfloor x \rfloor - \lfloor -x \rfloor = \begin{cases} 2\lceil x \rceil - 1 & \text{, if } x \notin Z \\ 2\lceil x \rceil & \text{, if } x \in Z \end{cases}$

12. $\log_a 11 = 0$, where $a > 0, a \neq 1$
13. $\log_a a = 1$, where $a > 0, a \neq 1$
14. $\log_a (xy) \log_a |x| + \log_a |y|$, where $a > 0, a \neq 1$ and $xy > 0$
15. $\log_a \left(\frac{x}{y}\right) \log_a |x| + \log_a |y|$, where $a > 0, a \neq 1$ and $\frac{x}{y} > 0$
16. $\log_a (x^n) = n \log_a |x|$, where $a > 0, a \neq 1$ and $x^n > 0$
17. $\log_a^n x^m = \frac{m}{n} \log_{|a|} |x|$, where $a > 0, a \neq 1$ and $x > 0$
18. $x^{\log_e y} = y^{\log_e x}$, where $x > 0, y > 0, a > 0, a \neq 1$
19. if $a > 1$, then the values of $f(x) = \log_a x$ increase with the increase in x i.e.

    $x < y \Leftrightarrow \log_a x < \log_a y$

Also,

$$\log_a x \begin{cases} < 0 \text{ for } 0 < x < 1 \\ = 0 \text{ for } x = 1 \\ > 0 \text{ for } x > 1 \end{cases}$$

20. If $0 < a < 1$, then the values of $f(x) = \log_a x$ decrease with the increase in x i.e.,

$x < y \Leftrightarrow \log_a x > \log_a y$

Also,

$$\log_a x \begin{cases} > 0 \text{ for } 0 < x < 1 \\ = 0 \text{ for } x = 1 \\ < 0 \text{ for } x > 1 \end{cases}$$

21. $\log_a x = \dfrac{1}{\log_x a}$ for $a > 0$, $a \neq 1$ and $x > 0$, $x \neq 1$.

22. If $f : D_1 \to R$ and $g : D_2 \to R$ are two real functions and $c \in R$, then
    (i) $f \pm g : D_1 \cap D_2 \to R$ is defined as $(f \pm g)(x) = f(x) \pm g(x)$
    (ii) $fg : D_1 \cap D_2 \to R$ is defined as $(fg)(x) = f(x)\, g(x)$
    (iii) $\left(\dfrac{f}{g}\right) : D_1 \cap D_2 - \{x : g(x) = 0\} \to R$ is deifined as $\left(\dfrac{f}{g}\right)(x) = \dfrac{f(x)}{g(x)}$
    (iv) $cf : D_1 \cap D_2 \to R$ is defined as $(cf)(x) = c\, f(x)$.

EXERCISE-1

1. Function as a special kind of relation.
2. Let f be a relation on the set N of natural numbers defined by $f = \{(n, 3n) : n \in N$. Is f a function from N to N. If so, find the range of f.
3. Function as a correspondence.
4. Domain, co–domain and range of a function.
5. Consider a rule $f(x) = 2x - 3$ associating elements of N (set of natural numbers) to elements of N. This rule does not define a function from N to itself, because $f(1) = 2 \times 1 - 3 = -1 \notin N$ i.e., $1 \in N$ is not associated to any elements.
6. Equal functions,?
7. Let $f : R - \{2\} \to R$ be defined by $f(x) = \dfrac{x^2 - 4}{x - 2}$ and $g : R \to R$ be defined by $g(x) = x + 2$. Find whether $f = g$ or not.
8. Let $f : z \to Z$, $g : z \to Z$ be function defined by $f = \{(n, n^2) : n \in Z\}$, $g = \{(n, |n|^2) : n \in Z\}$. Show that : $f = g$.
9. Express the following functions as set of ordered pairs and determine their ranges :
    (a) $f : A \to R$, $f(x)\ x^2 + 1$, where $A = \{-1, 0, 2, 4\}$.
    (b) $g : A \to N$, $g(x) = 2x$, where $A = \{x : x \in N, x \leq 10\}$.

10. Find the domain for which the function $f(x) = 2x^2 - 1$ and $g(x) = 1 - 3x$ are equal.
11. Given A = {–1, 0, 2, 5, 6, 11}, B = {–2, –1, 0, 18, 28, 108} and $f(x) = x^2 - x - 2$. Is f(A) = B ? Find f(A).
12. Let f : R → R be given by f(x) $x^2 + 3$. Find :
    (a) {x : f(x) = 28}
    (b) the pre–images of 39 and 2 under f.
13. Let f : R → R be a function given by $f(x) = x^2 + 1$. Find :
    (i) $f^{-1}\{-5\}$ (ii) $f^{-1}\{26\}$ (iii) $f^{-1}\{10, 37\}$
14. If f : R → R be defined as follows : $f(x) = \begin{cases} 1, \text{if } x \in Q \\ -1, \text{if } x \notin Q \end{cases}$

    Find :

    (a) f(1/2), $f(1/2), f(\pi), f\left(\sqrt{2}\right)$
    (b) Range of f
    (c) pre–image of 1 and –1.
15. Let f : R → R be such that $f(x) = 2^x$. Determine
    (a) Range of f
    (b) {x : f(x) = 1}
    (c) Whether f(x + y) = f(x) . f(y) holds.
16. Let A be the set of two positive integers. Let F : A → Z^+ (set of positive integers be defined by)

    f(n) = p, where p is the highest prime factor of n

    If range of f = {3}. Find set A. Is A unquely determined ?
17. Let A ⊆ N and f : A → A be defined by

    f(n) : the highest prime factor of n.

    If range of f is A. Determine A. Is A uniquely determined.
18. What is the fundamental difference between a relation and a function ? Is every relation a function ?
19. Let A = {–2, –1, 0, 1, 2} and f : A → Z be a function defined by $f(x) = x^2 - 2x - 3$. Find :
    (a) range of f i.e. f(A)
    (b) pre–images of 6, –3 and 5.
20. If a function f : R → R be defined by $f(x) = \begin{cases} 3x - 2 & , \quad x < 0; \\ 1 & , \quad x = 0; \\ 4x + 1 & , \quad x > 0. \end{cases}$

    Find f(1), f(–1), f(0), f(2).
21. A function f : R → R is defined by $f(x) = x^2$. Determine
    (a) range of f,

(b) $\{x : f(x) = 4\}$

(c) $\{y : f(y) = -1\}$.

22. Write the following relations as sets of ordered pairs and find which of them are functions :

(a) $\{(x, y) : y = 3x, x \in \{1,2, 3\}, y \in \{3, 6, 9, 13\}\}$

(b) $\{(x, y) : y > x + 1, x = 1, 2, \text{ and } y = 2, 4, 6\}$

(c) $\{(x, y) : x + y = 3, x, y \in \{0, 1, 2, 3\}\}$

23. If f, g, h are three functions defined from R to R as follows :

(i) $f(x) = x^2$

(ii) $g(x) = \sin x$

(iii) $h(x) = x^2 + 1$

Find the range of each function.

24. Let X = {1, 2, 3, 4} and Y = {1, 5, 9, 11, 15, 16}

Determine which of the following sets are functions from X to Y

(a) $f_1 = \{(1, 1), (2, 1), (3, 1), (4, 15)\}$

(b) $f_2 = \{(1, 1,), (2, 7), (3, 5)$

(c) $f_3 = \{(1, 5), (2, 9), (3, 1), (4, 5), (2, 11)\}$.

25. Let A = {12, 13, 14, 15, 16, 17} and $f : A \to Z$ be a function given by f(x) = highest prime factor of x.

Find range of f.

26. If $F : R \to R$ be defined by $f(x) = x^2 + 1$, then find $f^{-1}\{17\}$ and $f^{-1}\{-1\}$.

27. Let A = {9, 10, 11, 12, 13} and $f : A \to N$ be defined by f(n) = the highest prime factor of n. Find the range of f.

28. The function f is defined by $f(x) = \begin{cases} x^2 & , \quad 0 \le x \le 3 \\ 3x & , \quad 3 \le x \le 10 \end{cases}$

The relation g is defined by $g(x) = \begin{cases} x^2 & , \quad 0 \le x \le 2 \\ 3x & , \quad 2 \le x \le 10 \end{cases}$

Show that f is a function and g is not a function.

29. If $f(x) = x^2$, find $\dfrac{f(1.1) - f(1)}{(1.1) - 1}$

30. If $f(x) = 3x^4 - 5x^2 + 9$, find $f(x - 1)$.

31. If $f(x) = x + \dfrac{1}{x}$, prove that $[f(x)]^3 = f(x^3) + 3f\left(\dfrac{1}{x}\right)$.

32. If $f(x) = \dfrac{1}{2x+1}, x \ne -\dfrac{1}{2}$, then show that

$f(f(x)) = \dfrac{2x+1}{2x+3}$, prove that $x \neq -\dfrac{3}{2}$

33. If f(x) $\dfrac{x-1}{x+1} x \neq -1$, then show that $f(f(x)) = -\dfrac{1}{x}$, provided that $x \neq 0$.

34. Let f be defined by f(x) = x – 4 and g be defined by

$$g(x) = \begin{cases} \dfrac{x^2-16}{x+4} & , \quad x \neq -4 \\ \lambda & , \quad x = -4 \end{cases}$$

Find λ such that f(x) = g(x) for all x.

35. If f is a real function defined by $f(x) = \dfrac{x-1}{x+1}$, then prove that :

$$f(2x) = \frac{3f(x)+1}{f(x)+3}$$

36. If $f(x) = (x - a)^2 (x - b)^2$, find f(a + b).

37. If $y(x) = \dfrac{ax-b}{bx-a}$, show that x = f(y).

38. If $f(x) = \begin{cases} x^2 & , \quad \text{when } x < 0 \\ x & , \quad \text{when } 0 \le x < 1 \\ \dfrac{1}{x} & , \quad \text{when } x \ge 1 \end{cases}$

Find (a) f(1/2) (b) f(–2) (c) f(1) (d) $f(\sqrt{3})$

(e) $f\left(\sqrt{-3}\right)$

39. If $f(x) = x^3 - \dfrac{1}{x^3}$, show that $f(x) + f\left(\dfrac{1}{x}\right) = 0$.

40. If for non–zero x, $a f(x) + b f\left(\dfrac{1}{x}\right) = \dfrac{1}{x} - 5$, where $a \neq b$, then find f(x).

41. If $f(x) = \dfrac{x+1}{x-1}$, show that f[f{(x)}] = x.

42. Find the domain of each of the following functions :

(i) $f(x) = \sqrt{x-2}$

(ii) $f(x) = \dfrac{1}{\sqrt{1-x}}$

(iii) $f(x) = \sqrt{4-x^2}$

43. Find the domain of the function f(x) defined by

$$f(x) = \sqrt{4-x} + \frac{1}{\sqrt{x^2-1}}$$

44. Find the domain and range of the function f(x) given by

$$f(x) = \frac{x-2}{3-x}$$

45. Find the range of each of the following functions :

(i) $f(x) = \frac{1}{\sqrt{x-5}}$ (ii) $f(x) = \sqrt{16-x^2}$

(iii) $f(x) = \frac{x}{1+x^2}$ (iv) $f(x) = \frac{3}{2-x^2}$

46. Find the domain and range of the function $f(x) = \frac{x^2-9}{x-3}$

47. Let $f = \left\{\left(x, \frac{x^2}{1+x^2}\right) : x \in R\right\}$ be a function from R into R. Determine the range of f.

48. Find the domain and range of the function

$$f = \left\{\left(x : \frac{1}{1-x^2}\right) : x \in R, x \neq \pm 1\right\}$$

49. Find the domain and range of the function $f(x) = \frac{1}{2-\sin 3x}$

50. Find the domain of each of the following real valued functions of real variables :

(i) $f(x) = \frac{1}{x}$ (ii) $f(x) = \frac{1}{x-7}$ (iii) $f(x) = \frac{3x-2}{x+1}$

(iv) $f(x) = \frac{2x+1}{x^2-9}$ (v) $f(x) = \frac{x^2+2x+1}{x^2-8x+12}$

51. Find the domain of each of the following real valued functions for real variable :

(i) $f(x) = \sqrt{x-2}$ (ii) $f(x) = \frac{1}{\sqrt{x^2-1}}$

(iii) $f(x) = \sqrt{9-x^2}$ (iv) $f(x) = \sqrt{\frac{x-2}{3-x}}$

52. Fractional Part Function.

53. Exponential function.

54. Logarithmic function.

55. Reciprocal of a function.
56. What are the sum and difference of the identity function and the reciprocal function ?
57. Find the product of the identity function and the modulus function.
58. Find the quotient of the identity function by the modulus function.
59. Find the product of the identity function and the reciprocal function.
60. Find the quotient of the identity function by the reciprocal function.

61. Let C be a non–zero real number and f : R → R be a function defined by $f(x) = \frac{x}{c}$ for all $x \in R$.

    Find (i) cf (ii) c^2f (iii) $\left(\frac{1}{c}\right)f$.

62. Let f and g be real functions defined by $f(x) = \sqrt{x+2}$ and $g(x) = \sqrt{4-x^2}$. Then find each of the following functions :

    (i) f + g (ii) f – g (iii) fg

    (iv) $\frac{f}{g}$ (v) ff (vi) gg

63. Let f be the exponential function and g be the logarithmic function. Find :

    (i) (f + g) (1) (ii) (fg) (1) (iii) (3f) (1) (iv) (5 g) (1)

64. Find the domain of each of the following function given by :

    (i) $f(x) = \frac{1}{\sqrt{x-|x|}}$ (ii) $f(x) = \frac{1}{\sqrt{x+|x|}}$

    (iii) $f(x) = \frac{1}{\sqrt{x-[x]}}$ (iv) $\frac{1}{\sqrt{x+[x]}}$

65. Find the domain of definition of the function f(x) given by

    $f(x) = \log_4 \{\log_5 (\log_3(18x - x^2 - 77))\}$

66. Find the domain of definition of the function f(x) givey by

    $$f(x) = \frac{1}{\log_{10}(1-x)} + \sqrt{x+2}$$

67. Find the range of each of the following functions :

    (i) $f(x) = | x - 3 |$ (ii) $f(x) = 1 - | x - 2 |$ (iii) $f(x) = \frac{|x-4|}{x-4}$

68. Find the domain and range of each of the following functions given by :

    (i) $f(x) = \frac{1}{\sqrt{x-[x]}}$ (ii) $f(x) = 1 - | x - 3 |$

69. Find the domain of the real function f(x) defined by

$$f(x) = \sqrt{\frac{1-|x|}{2-|x|}}$$

70. Find f + g, f – g, cf = (c ∈ R, C ≠ 0), fg, $\frac{1}{f}$ and $\frac{f}{g}$ in each of the following :

(i) If f(x) $x^3 + 1$ and g(x) = x + 1

(ii) If f(x) = $\sqrt{x-1}$ and g(x) = $\sqrt{x+1}$

71. If f(x) = $\log_e(1 - x)$ and g(x) = [x], then determine each of the following functions :

(i) f + g (ii) fg (iii) $\frac{f}{g}$ (iv) $\frac{g}{f}$

Also, find (f + g) (–1), (fg) (0), $\left(\frac{f}{g}\right)\left(\frac{1}{2}\right), \left(\frac{g}{f}\right)\left(\frac{1}{2}\right)$.

72. If f, g, h are real function defined by f(x) = $\sqrt{x+1}$, g(x) = $\frac{1}{x}$ and h(x) = $2x^2 - 3$, then find the values of (2f + g – h) (1) and (2f + g – h) (0).

73. The function f is defined by $f(x) = \begin{cases} 1-x, & x<0 \\ 1, & x=0 \\ x+1, & x>0 \end{cases}$. Draw the graph of f(x).

EXERCISE-2

Answer each of the following questions in one word or one sentence or as per exact requirement of the questions.

1. Write the range of the real function f(x) = | x |.

2. Write the range of the function f(x) = sin [x], where $\frac{-\pi}{4} \le x \le \frac{\pi}{4}$

3. If f(x) = cos $[\pi^2]$ x + cos $[-\pi^2]$ x, where [x] denotes the greatest integer less than or equal to x, then write the value of f(π).

4. Write the range of the function f(x) $e^{x-[x]}$, x ∈ R.

5. If f(x) = 1 – $\frac{1}{x}$, then write the value of $f\left(f\left(\frac{1}{x}\right)\right)$.

6. If f(x) = $4x - x^2$, x ∈ R, then write the value of f(a + 1) – f(a – 1).

7. Write the domain and range of function f(x) given by f(x) = $\frac{1}{\sqrt{x-|x|}}$

8. Write the domain and range of function f(x) given by f(x) = $\sqrt{[x]-x}$.

EXERCISE-3

1. If f(x) = cos (log x), then

$$f(x^2)\, f(y^2) - \frac{1}{2}\left\{f\left(\frac{x^2}{y^2}\right) + f(x^2y^2)\right\}$$ has the value

(a) –2 (b) –1 (c) 1/2 (d) none of these

2. If f(x) = cos (log x), then $f(x)\, f(y) - \frac{1}{2}\left\{f\left(\frac{x}{y}\right) + f(xy)\right\}$ has the value

(a) –1 (b) 1/2 (c) –2 (d) none of these

3. The range of f(x) = cos [x], for π/2 < x < π/2 is

(a) {–1, 1, 0} (b) {cos 1, cos 2, 1} (c) {cos 1, – cos 1, 1} (d) [–1, 1]

4. Which of the following are functions :

(a) $\{(x, y) : y^2 = x, x, y \in R\}$ (b) $[(x, y(: y = |x|, x, y \in R\}$

(c) $\{(x, y) : x^2 + y^2 = 1, x, y \in R$ (d) $\{(x, y) : x^2 - y^2 = 1, x, y \in R\}$

5. If $f(x) = \log\left(\frac{1+x}{1-x}\right)$ and $g(x) = \frac{3x+3^3}{1+3x^2}$, then f(g(x) is equal to

(a) f(3x) (b) $\{f(x)\}^3$ (c) 3f(x) (d) –f(x)

6. If $f(x) = \log\left(\frac{1+x}{1-x}\right)$, then $f\left(\frac{2x}{1+x^2}\right)$ is equal to

(a) $\{f(x)\}^2$ (b) $\{f(x)\}^3$ (c) 2f(x) (d) 3f(x)

7. If $f(x) = \frac{2^x + 2^{-x}}{2}$, then f(x + y) f(x – y) is equals to

(a) $\frac{1}{2}$ {f(2x) + f(2y)] (b) $\frac{1}{2}$ [f(2x) – f(2y)] (c) $\frac{1}{4}$ [f(2x) + f(2y)] (d) $\frac{1}{4}$ [f(2x) – f(2y)]

8. If $2f(x) - 3f\left(\frac{1}{x}\right) = x^2$ (x ≠ 0), then f(2) is equal to

(a) $-\frac{7}{4}$ (b) $\frac{5}{2}$ (c) –1 (d) none of these

9. If $f(x) = \cos(\log_e x)$, then $f\left(\frac{1}{x}\right) f\left(\frac{1}{y}\right) - \frac{1}{2}\left\{f(xy) + f\left(\frac{x}{y}\right)\right\}$ is equal to

(a) cos(x – y) (b) log (cos (x – y)) (c) 1 (d) cos(x + y)

10. The function f : R → Ris defined by $f(x) = \cos^2 x + \sin^4 x$. Then, f(R) =

(a) [3/4, 1] (b) (3/4, 1] (c) [3/4, 1] (d) (3/4, 1)

11. If f : R → R and g : R → R are defined by f(x) = 2x + 3 and g(x) $x^2 + 7$, then the values of x such that g(f(x)) = 8 are
(a) 1, 2 (b) –1, 2 (c) –1, –2 (d) 1, –2

12. If f : [–2, 2] → R is defined by

$$f(x) = \begin{cases} -1 & ,\text{for } -2 \le x \le 0 \\ x-1 & ,\text{for } 0 \le x \le 2 \end{cases}, \text{ then}$$

$\{x \in [-2, 2] : x \le 0 \text{ and } f(|x|) = x\} =$

(a) {–1} (b) {0} (c) $\left\{-\frac{1}{2}\right\}$ (d) ϕ

13. If $f(x) = 64x^3 + \frac{1}{x^3}$ and α, β are the roots of $4x + \frac{1}{x} = 3$. Then,
(a) f(α) ≠ f(β) = –9 (b) f(α) = f(β) = 63 (c) f(α) ≠ f(β) (d) none of these

14. If $e^{f(x)} = \frac{10+x}{10-x}, x \in (-10, 10)$ and f(x) $kf\left(\frac{200x}{100+x^2}\right)$, then k =
(a) 0.5 (b) 0.6 (c) 0.7 (d) 0.8

15. If f : R → R be given by

$$f(x) = \frac{4^x}{4^x + 2} \text{ for all } x \in R. \text{ Then}$$

(a) f(x) = f(1 – x) (b) f(x) + f(1 – x) = 0 (c) f(x) + f(1 – x) = 1 (d) f(x) + f(x – 1) =1

16. The domain of definition of $f(x) = \sqrt{\frac{x+3}{(2-x)(x-5)}}$ is
(a) (–∞, –3] ∪ (2, 5) (b) (–∞, –3) ∪ (2, 5) (c) (–∞, –3] ∪ [2, 5] (d) none of these

17. The domain of definition of the function $f(x) = \sqrt{\frac{x-2}{x+2}} + \sqrt{\frac{1-x}{1+x}}$ is
(a) (–∞, –2) ∪ (2, ∞) (b) [–1, 1] (c) ∞ (d) none of these

18. The domain of definition of $f(x) = \sqrt{4x - x^2}$ is
(a) R – [0, 4] (b) R – (0, 4) (c) (0, 4) (d) [0, 4]

19. The domain of definition of $f(x) = \sqrt{x-3-2\sqrt{x-4}} - \sqrt{x-3+2\sqrt{x-4}}$ is
(a) [4, ∞) (b) (–∞, 4] (c) (4, ∞) (d) (–∞, 4)

20. The range of the function f(x) = | x – 1 | is
(a) (–∞, 0) (b) [0, ∞) (c) (0, ∞) (d) R