3

MATRICES

IMPORTANT FORMULAE

1. **Matrix :** A matrix is defined as a rectangular array (arrangement) of number or functions.
2. **Order of matrix :** A matrix, having m rows and n columns is called of order $m \times n$ or $m \times n$ matrix.
3. **Types of Matrices :**

- **Square Matrix :** If in a matrix, the number of rows is equal to the number of columns then the matrix is called square matrix. For example :

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}, \begin{bmatrix} 3 & 4 \\ 6 & 7 \end{bmatrix}, \begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix}$$

are 2×2, 2×2 and 3×3 square matrices.

- **Row Matrix or Row Vector :** If a matrix has only one row, the matrix is called a row matrix or row vector. For example : [3 4], [2 4 6], [1 3 7 8] and $[a_1\ a_2\ .....\ a_n]$ are respectively 1×2, 1×3, 1×4 and $1 \times n$ order matrices.
- **Column Matrix :** If a matrix has only one column, the matrix is called a column matrix. For example :

$$B = \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix}$$

Here B is a column matrix. Its order is 4×1.

- **Zero Matrix or Null Matrix :** If all the elements of a matrix are zero, the matrix is called a null matrix. It is denoted by 0. For Example :

$$O_{2\times3} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, O_{3\times3} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

- **Diagonal Matrix :** A square matrix is said to be a diagonal matrix if all its elements except the diagonal elements are zero. For example :

$$A = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 4 \end{bmatrix}$$ is a diagonal matrix.

- **Scalar Matrix :** If all the elements of a diagonal matrix are equal, the matrix is called a scalar matrix.

$$a_{ij} = \begin{cases} k, & \text{if } i = j \\ 0, & \text{if } i \neq j \end{cases}$$

For example : $A = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}, B = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

Here A and B are scalar matrices.

- **Unit or Identity Matrix :** If all the elements of a diagonal matrix are equal to unity, the matrix is called a unit matrix. For example :

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}_{2\times2} \text{ and } \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}_{3\times3}$$

are respectively 2×2 and 3×3 unit matrices and denoted by I_2 and I_3.

Properties of Matrix Addition :

- **Commutative law for addition :** If A and B are two matrices of the same order, then

$$A + B = B + A$$

- **Associative law for addition :** If A, B and C are three matrices of the same order, then

$$A + (B + C) = (A + B) + C.$$

- **Existence of additive identity :** If $A = [a_{ij}]$ be an $m \times n$ matrix, then $A + O = A = O + A$, where O is a zero matrix of order $m \times n$.
- **Existence of additive inverse :** If $A = [a_{ij}]$ is an $m \times n$ matrix, then there exists a unique matrix

$$-A = [-a_{ij}] \text{ such that}$$

$$A + (-A) = (-A) + A = O, \text{ where O is a zero matrix.}$$

- **Multiplication of a matrix by a scalar :** Let A be a $m \times n$ matrix and k is a scalar (number). Multiplication by k of matrix A is denoted by kA which is such that if $A = [a_{ij}]_{m\times n}$ then $kA = [ka_{ij}]_{m\times n}$.
- $1.A = A$
- $m(nA) = (mn)A = n(mA)$
- $(m + n)A = mA + nA$
- $m(A + B) = mA + mB$
- **Multiplication of Matrices :** Let $A = [a_{ij}]_{m\times n}$ and $B = [b_{jk}]_{n\times p}$ be two matrices such that number of columns of A is equal to the number of rows of B, then

$$\text{matrix } C = [c_{ik}]_{m\times p},$$

where $c_{jk} = \sum_{j=1}^{n} a_{ij}b_{jk} = a_{i1}b_{1k} + a_{i2}b_{2k}\, a_{i3}b_{3k} + ... + a_{in}b_{nk}$ is called the product of matrices A and B and denoted by AB.

Note : (1) Product of matrices A and B is defined if and only if, the number of columns in A is equal to the number of rows in B.

(2) If A is matrix of $m \times n$ order and B is matrix of $n \times p$ order, then AB is the matrix of order $m \times p$.

(3) If the elements of ith row of A are :

$$a_{i1}\, a_{i2}\, a_{i3}\, ...\, a_{in} \text{ and}$$

the elements of kth column of B are :

$b_{1k}\, a_{2k}\, a_{3k}\, ...\, b_{nk}$, then the product of the elements of ith row of A and kth column of B is :

$$[a_{i1}\, b_{i2} \ldots a_{in}] \begin{bmatrix} b_{ik} \\ b_{2k} \\ \vdots \\ b_{nk} \end{bmatrix} = a_{i1}b_{1k} + a_{i2}b_{2k} + a_{i3}b_{3k} + \ldots + a_{in}b_{nk}$$

Properties : (1) In the product AB, A is called pre-factor and B is called post-factor.

(2) It is not necessary that if AB is defined then BA is also defined.

(3) If A is a $m \times n$ matrix and AB and BA both are defined, then B will a $m \times n$ matrix.

(4) For same order square matrices A and B, AB and BA both are defined.

(5) It is not necessary that AB = BA.

Transpose of a Matrix

A matrix obtained by interchanging the rows and columns of the matrix A is called the transposed matrix of A and is denoted by A′ or A^T.

Thus is $A = [a_{ij}]$, then $A' = [a_{ji}]$

For example :

$$A = \begin{bmatrix} 3 & 3 & 2 \\ 5 & 2 & 3 \end{bmatrix}_{2\times 3} \quad \text{then } A' = \begin{bmatrix} 3 & 5 \\ 3 & 2 \\ 2 & 3 \end{bmatrix}_{3\times 2}$$

- If A′ = A, then A is a symmetric matrix.
- If A′ = – A, then A is a skew-symmetric matrix.

Symbols to Denote Different Elementary Transformations :

- Interchanging any two rows (or columns). This transformation is indicated by $R_i \Leftrightarrow R_j$ ($C_i \Leftrightarrow C_j$).
- Multiplication of the elements of any row (or column) by a non-zero scalar quantity is indicated by $R_1 \Leftrightarrow kR_i$ ($C_1 \Leftrightarrow kC_i$).
- Addition of constant multiple of the elements of any row to the corresponding elements of any other row is indicated by $R_i \Leftrightarrow R_i + kR_j$ ($C_i \Leftrightarrow kC_j$).

Invertible Matrices

If A is a square matrix of order m, and if there exists another square matrix B of the same order m such that AB = BA = I, then B is called the inverse matrix of A and it is denoted by A^{-1}. The matrix A is called an invertible and non-singular matrix.

➠ Multiple Choice Questions

1. $A = [a_{ij}]_{m\times n}$ is a square matrix if : (*BSEB*, 2012)
(a) $m = n$ (b) $m < n$
(c) $m > n$ (d) none of these

2. If A and B are square matrices, then (AB)′ =
(a) B′A′ (b) A′B′
(c) AB′ (d) A′B (*BSEB*, 2012)

3. Let A be a non-singular matrix of order $n \times n$, then [adj A] =
(a) $n\,|A|$ (b) $|A|^{n-1}$
(c) $|A|$ (d) $|A|^n$

4. If $A = \begin{bmatrix} \alpha & 0 \\ 1 & 1 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 0 \\ 5 & 1 \end{bmatrix}$, where $A^2 = B$, then the value of α is : (*BSEB*, 2010)
(a) 1 (b) –1
(c) 4 (d) no real value of α

5. The inverse of $\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$ is : (*BSEB*, 2010)
(a) $\begin{bmatrix} -\cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$ (b) $\begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$
(c) $\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & -\cos\theta \end{bmatrix}$ (d) none of these

6. If $A = \begin{bmatrix} i & 0 \\ 0 & i \end{bmatrix}$, then A^2 = (*BSEB*, 2011)
(a) $\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$ (b) $\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$
(c) $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ (d) $\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$

7. A matrix $A = [a_{ij}]_{n\times n}$ is said to be symmetric if : (*BSEB*, 2013)
(a) $a_{ij} = 0$ (b) $a_{ij} = -a_{ji}$
(c) $a_{ij} = a_{ji}$ (d) $a_{ij} = 1$

8. A matrix $A = [a_{ij}]_{n\times n}$ is said to be skew-symmetric if : (*BSEB*, 2013)
(a) $a_{ij} = 0$ (b) $a_{ij} = a_{ji}$
(c) $a_{ij} = -a_{ji}$ (d) $a_{ij} = 1$

9. If $A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$ then A^2 is : (*BSEB*, 2013)
(a) 27A (b) 2A (c) 3A (d) I

10. If I is a unit matrix of order 2 × 2, then the value of |I| is :
(a) 0 (b) 1 (c) 2 (d) – 1

11. The values of a and b, when :
$\begin{bmatrix} a & b \\ -a & 2b \end{bmatrix}\begin{bmatrix} 2 \\ -1 \end{bmatrix} = \begin{bmatrix} 5 \\ 4 \end{bmatrix}$ are : (*JAC*, 2013)
(a) $a = 1, b = -3$ (b) $a = -1, b = 3$
(c) $a = 1, b = 3$ (d) $a = -1, b = -3$

12. If $2\begin{bmatrix} x & 5 \\ 3 & y \end{bmatrix} = \begin{bmatrix} 4 & 10 \\ 6 & 6 \end{bmatrix}$, then the values of x and y are : (*USEB*, 2013)
(a) $x = 2, y = 3$ (b) $x = 3, y = 2$
(c) $x = 2, y = 2$ (d) $x = 3, y = 3$

13. If $\begin{bmatrix} 2x-y & 5 \\ 3 & y \end{bmatrix} = \begin{bmatrix} 6 & 5 \\ 3 & -2 \end{bmatrix}$, then x = (*BSEB*, 2015)
(a) 3 (b) 4 (c) 5 (d) 2

14. The inverse of $A = \begin{bmatrix} 2 & 3 \\ 5 & k \end{bmatrix}$ will not be obtained if k has the value : (*BSEB*, 2015)

(a) 2 (b) $\frac{3}{2}$ (c) $\frac{5}{2}$ (d) $\frac{15}{2}$

15. For any unit matrix I : (*BSEB*, 2015)
(a) $I^2 = I$ (b) $|I| = 0$ (c) $|I| = 2$ (d) $|I| = 5$

Ans. 1. (a), 2. (a), 3. (b), 4. (d), 5. (b), 6. (b), 7. (c), 8. (c), 9. (c), 10. (b), 11. (a), 12. (a), 13. (d), 14. (d), 15. (a).

➠ Very Short Answer Type Questions

Q. 1. If $A = \begin{bmatrix} 9 & 10 & 11 \\ 12 & 13 & 14 \end{bmatrix}$ and $B = \begin{bmatrix} 11 & 10 & 9 \\ 8 & 7 & 6 \end{bmatrix}$, then find A + B. (*BSEB*, 2014)

Solution : $A + B = \begin{bmatrix} 9 & 10 & 11 \\ 12 & 13 & 14 \end{bmatrix} + \begin{bmatrix} 11 & 10 & 9 \\ 8 & 7 & 6 \end{bmatrix}$

$= \begin{bmatrix} 9+11 & 10+10 & 11+9 \\ 12+8 & 13+7 & 14+6 \end{bmatrix}$

$= \begin{bmatrix} 20 & 20 & 20 \\ 20 & 20 & 20 \end{bmatrix}$

Q. 2. If $A = \begin{bmatrix} 2 & 7 \\ 9 & 8 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 2 \\ 0 & 3 \end{bmatrix}$, then find A – B. (*JAC*, 2013)

Solution : $A - B = \begin{bmatrix} 2 & 7 \\ 9 & 8 \end{bmatrix} - \begin{bmatrix} 1 & 2 \\ 0 & 3 \end{bmatrix}$

$= \begin{bmatrix} 2-1 & 7-2 \\ 9-0 & 8-3 \end{bmatrix}$

$= \begin{bmatrix} 1 & 5 \\ 9 & 5 \end{bmatrix}$

Q. 3. If $\begin{bmatrix} 9 & -1 & 4 \\ -2 & 1 & 3 \end{bmatrix} = A + \begin{bmatrix} 1 & 2 & -1 \\ 0 & 4 & 9 \end{bmatrix}$, then find the matrix A. (*CBSE*, 2013)

Solution : We have

$\begin{bmatrix} 9 & -1 & 4 \\ -2 & 1 & 3 \end{bmatrix} = A + \begin{bmatrix} 1 & 2 & -1 \\ 0 & 4 & 9 \end{bmatrix}$

$\Rightarrow \quad A = \begin{bmatrix} 9 & -1 & 4 \\ -2 & 1 & 3 \end{bmatrix} - \begin{bmatrix} 1 & 2 & -1 \\ 0 & 4 & 9 \end{bmatrix}$

$= \begin{bmatrix} 9-1 & -1-2 & 4+1 \\ -2-0 & 1-4 & 3-9 \end{bmatrix}$

$= \begin{bmatrix} 8 & -3 & 5 \\ -2 & -3 & -6 \end{bmatrix}$

Q. 4. Construct a 2 × 2 matrix whose $(i, j)^{th}$ element is given by $a_{ij} = \frac{2i - j}{3}$. (*JAC*, 2014)

Solution : Let $A = [a_{ij}]_{2 \times 2}$

Then $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}_{2 \times 2}$ $(i, j)^{th}$

Now $a_{ij} = \frac{2i - j}{3}$

$\therefore \quad a_{11} = \frac{2(1) - 1}{3} = \frac{1}{3}$

$a_{12} = \frac{2(1) - 2}{3} = 0$

$a_{21} = \frac{2(2) - 1}{3} = 1$

$a_{22} = \frac{2(2) - 2}{3} = \frac{2}{3}$

$\therefore \quad A = \begin{bmatrix} \frac{1}{3} & 0 \\ 1 & \frac{2}{3} \end{bmatrix}$

Q. 5. The elements a_{ij} of a 3 × 3 matrix are given by $a_{ij} = \frac{1}{2} |-3i + j|$. Write the value of element a_{32}. [*AI CBSE*, 2014 (*Comptt.*)]

Solution : We have

$a_{ij} = \frac{1}{2} |-3i + j|$

Put $i = 3$ and $j = 2$, we get

$a_{32} = \frac{1}{2} |-3(3) + 2|$

$= \frac{1}{2} |-9 + 2|$

$= \frac{1}{2} |-7|$

$= \frac{1}{2} (7)$

$= \frac{7}{2}$

Q. 6. If $A = \begin{bmatrix} 5 & -1 \\ 6 & 7 \end{bmatrix}$ and $B = \begin{bmatrix} 2 & 1 \\ 3 & 4 \end{bmatrix}$, then find AB. (*BSEB*, 2014)

Solution : $AB = \begin{bmatrix} 5 & -1 \\ 6 & 7 \end{bmatrix} \begin{bmatrix} 2 & 1 \\ 3 & 4 \end{bmatrix}$

$= \begin{bmatrix} 10-3 & 5-4 \\ 12+21 & 6+28 \end{bmatrix}$

$= \begin{bmatrix} 7 & 1 \\ 33 & 34 \end{bmatrix}$

Q. 7. Simplify :

$\cos\theta \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} + \sin\theta \begin{bmatrix} \sin\theta & -\cos\theta \\ \cos\theta & \sin\theta \end{bmatrix}$ (*CBSE Delhi*, 2012)

Solution : Given

$\cos\theta \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} + \sin\theta \begin{bmatrix} \sin\theta & -\cos\theta \\ \cos\theta & \sin\theta \end{bmatrix}$

$= \begin{bmatrix} \cos^2\theta & \sin\theta\cos\theta \\ -\sin\theta\cos\theta & \cos^2\theta \end{bmatrix} + \begin{bmatrix} \sin^2\theta & -\sin\theta\cos\theta \\ \sin\theta\cos\theta & \sin^2\theta \end{bmatrix}$

$= \begin{bmatrix} \cos^2\theta + \sin^2\theta & 0 \\ 0 & \cos^2\theta + \sin^2\theta \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

Q. 8. Find the value of $x + y$ from the following equation : $2\begin{bmatrix} x & 5 \\ 7 & y-3 \end{bmatrix} + \begin{bmatrix} 3 & -4 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 7 & 6 \\ 15 & 14 \end{bmatrix}$

(*CBSE Delhi*, 2012)

Solution : We have

$$\begin{bmatrix} x & 5 \\ 7 & y-3 \end{bmatrix} + \begin{bmatrix} 3 & -4 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 7 & 6 \\ 15 & 14 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 2x & 10 \\ 14 & 2y-6 \end{bmatrix} + \begin{bmatrix} 3 & -4 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 7 & 6 \\ 15 & 14 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 2x+3 & 6 \\ 15 & 2y-4 \end{bmatrix} = \begin{bmatrix} 7 & 6 \\ 15 & 14 \end{bmatrix}$$

Therefore, $2x + 3 = 7$

$\Rightarrow$ $2x = 4$

$\therefore$ $x = 2$

and $2y - 4 = 14$

$\Rightarrow$ $2y = 18$

$\therefore$ $y = 9$

Hence, $x + y = 2 + 9 = 11$

Q. 9. If $A^T = \begin{bmatrix} 3 & 4 \\ -1 & 2 \\ 0 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} -1 & 2 & 1 \\ 1 & 2 & 3 \end{bmatrix}$, then find $A^T - B^T$.

(*CBSE, Outside Delhi*, 2012)

Solution : Given,

$$A^T = \begin{bmatrix} 3 & 4 \\ -1 & 2 \\ 0 & 1 \end{bmatrix}$$

and $$B = \begin{bmatrix} -1 & 2 & 1 \\ 1 & 2 & 3 \end{bmatrix}$$

$\therefore$ $$B^T = \begin{bmatrix} -1 & 1 \\ 2 & 2 \\ 1 & 3 \end{bmatrix}$$

Therefore, $$A^T - B^T = \begin{bmatrix} 3 & 4 \\ -1 & 2 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} -1 & 1 \\ 2 & 2 \\ 1 & 3 \end{bmatrix}$$

$$= \begin{bmatrix} 4 & 3 \\ -3 & 0 \\ -1 & -2 \end{bmatrix}$$

Q. 10. Find the value of $y + x$ from the following equation :

$$2\begin{bmatrix} 1 & 3 \\ 0 & x \end{bmatrix} + \begin{bmatrix} y & 0 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 5 & 6 \\ 1 & 8 \end{bmatrix}$$

(*CBSE, Outside Delhi*, 2013)

Solution : We have

$$2\begin{bmatrix} 1 & 3 \\ 0 & x \end{bmatrix} + \begin{bmatrix} y & 0 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 5 & 6 \\ 1 & 8 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 2 & 6 \\ 0 & 2x \end{bmatrix} + \begin{bmatrix} y & 0 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 5 & 6 \\ 1 & 8 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 2+y & 6 \\ 1 & 2x+2 \end{bmatrix} = \begin{bmatrix} 5 & 6 \\ 1 & 8 \end{bmatrix}$$

Therefore, $2 + y = 5 \Rightarrow y = 3$

and $2x + 2 = 8$

$\Rightarrow$ $2x = 6 \Rightarrow x = 3$

Hence, $x + y = 3 + 3 = 6$

Q. 11. If $A = \begin{bmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{bmatrix}$, then find A^2.

(*RSEB*, 2013)

Solution : $A^2 = AA$

$$= \begin{bmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{bmatrix}\begin{bmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{bmatrix}$$

$$= \begin{bmatrix} \cos^2\alpha - \sin^2\alpha & \cos\alpha\sin\alpha + \sin\alpha\cos\alpha \\ -\sin\alpha\cos\alpha - \cos\alpha\sin\alpha & -\sin^2\alpha + \cos^2\alpha \end{bmatrix}$$

$$= \begin{bmatrix} \cos 2\alpha & \sin 2\alpha \\ -\sin 2\alpha & \cos 2\alpha \end{bmatrix}$$

Q. 12. If $A = \begin{bmatrix} -2 \\ 4 \\ 5 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 4 & -6 \end{bmatrix}$, then find AB.

(*RSEB*, 2014)

Solution :

$$AB = \begin{bmatrix} -2 \\ 4 \\ 5 \end{bmatrix}_{3\times 1} \begin{bmatrix} 1 & 4 & -6 \end{bmatrix}_{1\times 3}$$

$$= \begin{bmatrix} -2 & -8 & 12 \\ 4 & 16 & -24 \\ 5 & 20 & -30 \end{bmatrix}_{3\times 3}$$

Q. 13. Find the transpore of the matrix $\begin{bmatrix} 5 & \frac{1}{2} & -1 \end{bmatrix}$.

(*USEB*, 2014)

Solution : Let $A = \begin{bmatrix} 5 & \frac{1}{2} & -1 \end{bmatrix}$,

Then $$A' = \begin{bmatrix} 5 & \frac{1}{2} & -1 \end{bmatrix}'$$

$$= \begin{bmatrix} 5 \\ \frac{1}{2} \\ -1 \end{bmatrix}$$

Q. 14. If $A = \begin{bmatrix} 1 & 5 \\ 6 & 7 \end{bmatrix}$, then find $A + A'$. (*USEB*, 2013)

Solution : $$A = \begin{bmatrix} 1 & 5 \\ 6 & 7 \end{bmatrix}$$

$\therefore$ $$A' = \begin{bmatrix} 1 & 5 \\ 6 & 7 \end{bmatrix}' = \begin{bmatrix} 1 & 6 \\ 5 & 7 \end{bmatrix}$$

$\therefore$ $$A + A' = \begin{bmatrix} 1 & 5 \\ 6 & 7 \end{bmatrix} + \begin{bmatrix} 1 & 6 \\ 5 & 7 \end{bmatrix}$$

$$= \begin{bmatrix} 1+1 & 5+6 \\ 6+5 & 7+7 \end{bmatrix}$$

$$= \begin{bmatrix} 2 & 11 \\ 11 & 14 \end{bmatrix}$$

Q. 15. For what value of x, is the matrix

$$A = \begin{bmatrix} 0 & 1 & -2 \\ -1 & 0 & 3 \\ x & -3 & 0 \end{bmatrix}$$ **a skew-symmetric matrix ?**

(AI CBSE, 2013)

Solution : For a skew-symmetric matrix,

$$a_{ij} = -a_{ji}$$

Here, $a_{31} = x$

and $a_{13} = -2$

$\therefore$ $a_{31} = -a_{13}$

$\Rightarrow$ $x = -(-2) = 2$

Q. 16. If A_{ij} is the co-factor of the element a_{ij} of the determinant $\begin{vmatrix} 2 & -3 & 5 \\ 6 & 0 & 4 \\ 1 & 5 & -7 \end{vmatrix}$**, then write the value of a_{32} A_{32}.** *(AI CBSE, 2013)*

Solution : $a_{32} = 5$

$$A_{32} = (-1)^{3+2} \begin{vmatrix} 2 & 5 \\ 6 & 4 \end{vmatrix}$$

$$= -(8 - 30) = 22$$

$\therefore$ $a_{32} A_{32} = 5 \times 22 = 110$

Q. 17. Find adj A, if A = $\begin{bmatrix} 5 & 2 \\ 7 & 3 \end{bmatrix}$**.**

[CBSE, 2014 (Comptt.)]

Solution : $A = \begin{bmatrix} 5 & 2 \\ 7 & 3 \end{bmatrix}$

co-factor of 5 = 3

co-factor of 2 = – 7

co-factor of 7 = – 2

co-factor of 3 = 5

$$\therefore \quad \text{adj } A = \begin{bmatrix} 3 & -7 \\ -2 & 5 \end{bmatrix}'$$

$$= \begin{bmatrix} 3 & -2 \\ -7 & 5 \end{bmatrix}$$

Q. 18. For what value of x, the given matrix

$$A = \begin{bmatrix} 3 - 2x & x + 1 \\ 2 & 4 \end{bmatrix}$$ **is a singular matrix ?**

[CBSE, 2013 (Comptt.)]

Solution : If A is a singular matrix, then

$$|A| = 0$$

$$\Rightarrow \begin{vmatrix} 3 - 2x & x + 1 \\ 2 & 4 \end{vmatrix} = 0$$

$\Rightarrow$ $(3 - 2x) 4 - 2(x + 1) = 0$

$\Rightarrow$ $12 - 8x - 2x - 2 = 0$

$\Rightarrow$ $-10x + 10 = 0$

$\Rightarrow$ $10x = 10$

$\therefore$ $x = 1$

Q. 19. If A is an invertible square matrix of order 3 and |A| = 5, then find the value of |adj A|.

[CBSE, 2013 (Comptt.)]

Solution : $n = 3$

$|A| = 5$

We know that

$$|\text{adj } A| = |A|^{n-1}$$

$\therefore$ $|\text{adj } A| = 5^{3-1} = 5^2 = 25$

➠ Short Answer Type Questions

Q. 1. Find the values of a and b if

$$\begin{bmatrix} a - b & 2a + c \\ 2a - b & 3c + d \end{bmatrix} = \begin{bmatrix} -1 & 5 \\ 0 & 13 \end{bmatrix}$$ *(CBSE, 2013)*

Solution : We have

$$\begin{bmatrix} a - b & 2a + c \\ 2a - b & 3c + d \end{bmatrix} = \begin{bmatrix} -1 & 5 \\ 0 & 13 \end{bmatrix}$$

$\Rightarrow$ $a - b = -1$ ...(1)

$2a + c = 5$ ...(2)

$2a - b = 0$ ...(3)

$3c + d = 13$ ...(4)

(By definition of equality of two matrices)

Subtracting equation (1) from equation (3), we get

$$a = 1$$

Putting $a = 1$ in equation (1), we get

$$1 - b = -1$$

$\Rightarrow$ $b = 2$

Q. 2. If $\begin{bmatrix} x - y & 2y \\ 2y + z & x + y \end{bmatrix} = \begin{bmatrix} 1 & 4 \\ 9 & 5 \end{bmatrix}$**, then write the value of $x + y + z$.** *[CBSE, 2013 (Comptt.)]*

Solution : We have

$$\begin{bmatrix} x - y & 2y \\ 2y + z & x + y \end{bmatrix} = \begin{bmatrix} 1 & 4 \\ 9 & 5 \end{bmatrix}$$

$\Rightarrow$ $x - y = 1$ ...(1)

$2y = 4$ ...(2)

$2y + z = 9$ ...(3)

$x + y = 5$ ...(4)

(By definition of equality of two matrices)

Solving (1) and (4), we get

$$x = 3, y = 2$$

From (2), $y = 2$

Putting $y = 2$ in (3), we get

$$2(2) + z = 9$$

$\Rightarrow$ $4 + z = 9$

$\Rightarrow$ $z = 5$

$\therefore$ $x + y + z = 3 + 2 + 5$

$= 10$

Q. 3. If $2\begin{bmatrix} x & 5 \\ 3 & y \end{bmatrix} = \begin{bmatrix} 4 & 10 \\ 6 & 6 \end{bmatrix}$ **then find the values of x and y.** *(USEB, 2013)*

Solution : We have

$$2\begin{bmatrix} x & 5 \\ 3 & y \end{bmatrix} = \begin{bmatrix} 4 & 10 \\ 6 & 6 \end{bmatrix}$$

$\Rightarrow \begin{bmatrix} 2x & 10 \\ 6 & 2y \end{bmatrix} = \begin{bmatrix} 4 & 10 \\ 6 & 6 \end{bmatrix}$

$\Rightarrow \quad 2x = 4, 2y = 6$ (By definition of equality of two matrices)

$\Rightarrow \quad x = 2, x = 3$

Q. 4. Find the values of a and b for which

$$\begin{bmatrix} a & b \\ -a & 2b \end{bmatrix}\begin{bmatrix} 2 \\ -1 \end{bmatrix} = \begin{bmatrix} 5 \\ 4 \end{bmatrix}$$ (*JAC*, 2014)

Solution :

$\Rightarrow \begin{bmatrix} a & b \\ -a & 2b \end{bmatrix}\begin{bmatrix} 2 \\ -1 \end{bmatrix} = \begin{bmatrix} 5 \\ 4 \end{bmatrix}$

$\Rightarrow \quad 2a - b = 5 \quad ...(1)$
$\quad -2a - 2b = 4 \quad ...(2)$ (By definition of equality of two matrices)

Solving (1) and (2), we get

$a = 1, b = -3$

Q. 5. If $A + B = \begin{bmatrix} 5 & 2 \\ 0 & 9 \end{bmatrix}$ and $A - B = \begin{bmatrix} -3 & -6 \\ 4 & -1 \end{bmatrix}$, then find A. (*BSEB*, 2014)

Solution : We have

$$A + B = \begin{bmatrix} 5 & 2 \\ 0 & 9 \end{bmatrix} \quad ...(1)$$

and $$A - B = \begin{bmatrix} -3 & -6 \\ 4 & -1 \end{bmatrix} \quad ...(2)$$

Adding (1) and (2), we get

$$2A = \begin{bmatrix} 5 & 2 \\ 0 & 9 \end{bmatrix} + \begin{bmatrix} -3 & -6 \\ 4 & -1 \end{bmatrix}$$

$$= \begin{bmatrix} 5-3 & 2-6 \\ 0+4 & 9-1 \end{bmatrix}$$

$$= \begin{bmatrix} 2 & -4 \\ 4 & 8 \end{bmatrix}$$

$$\Rightarrow \quad A = \frac{1}{2}\begin{bmatrix} 2 & -4 \\ 4 & 8 \end{bmatrix}$$

$$\Rightarrow \quad A = \begin{bmatrix} 1 & -2 \\ 2 & 4 \end{bmatrix}$$

Q. 6. Solve the following matrix equation for x :

$$\begin{bmatrix} x & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ -2 & 0 \end{bmatrix} = 0$$ [*CBSE*, 2014 (*Comptt.*)]

Solution : We have

$$\begin{bmatrix} x & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ -2 & 0 \end{bmatrix} = 0$$

$\Rightarrow \quad \begin{bmatrix} x-2 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \end{bmatrix}$

$\Rightarrow \quad x - 2 = 0$ (By definition of equality of two matrices)

$\Rightarrow \quad x = 2$

Q. 7. If $2\begin{bmatrix} 3 & 4 \\ 5 & x \end{bmatrix} + \begin{bmatrix} 1 & y \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 7 & 0 \\ 10 & 5 \end{bmatrix}$, then find $(x - y)$ [*CBSE*, 2014 (*Comptt.*)]

Solution : We have

$$2\begin{bmatrix} 3 & 4 \\ 5 & x \end{bmatrix} + \begin{bmatrix} 1 & y \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 7 & 0 \\ 10 & 5 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 6 & 8 \\ 10 & 2x \end{bmatrix} + \begin{bmatrix} 1 & y \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 7 & 0 \\ 10 & 5 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 7 & 8+y \\ 10 & 2x+1 \end{bmatrix} = \begin{bmatrix} 7 & 0 \\ 10 & 5 \end{bmatrix}$$

$\Rightarrow \quad 8 + y = 0$
$\quad 2x + 1 = 5$ (By definition of equality of two matrices)

$\Rightarrow \quad x = 2; y = -8$

$\therefore \quad x - y = 2 - (-8) = 2 + 8 = 10$

Q. 8. If $\begin{bmatrix} 2x & 4 \end{bmatrix}\begin{bmatrix} x \\ -8 \end{bmatrix} = 0$, find the positive value of x. [*AI CBSE*, 2014 (*Comptt.*)]

Solution : We have

$$\begin{bmatrix} 2x & 4 \end{bmatrix}\begin{bmatrix} x \\ -8 \end{bmatrix} = 0$$

$\Rightarrow \quad [2x(x) + 4(-8)] = 0$

$\Rightarrow \quad [2x^2 - 32] = [0]$

$\Rightarrow \quad 2x^2 - 32 = 0$

$\Rightarrow \quad 2x^2 = 32$

$\Rightarrow \quad x^2 = 16$

$\Rightarrow \quad x = \pm 4$

Q. 9. If $\begin{bmatrix} x-y & z \\ 2x-y & w \end{bmatrix} = \begin{bmatrix} -1 & 4 \\ 0 & 5 \end{bmatrix}$, find the value of $x + y$. (*AI CBSE*, 2014)

Solution : We have

$$\begin{bmatrix} x-y & z \\ 2x-y & w \end{bmatrix} = \begin{bmatrix} -1 & 4 \\ 0 & 5 \end{bmatrix}$$

$\Rightarrow \quad x - y = -1 \quad ...(1)$
$\quad 2x - y = 0 \quad ...(2)$
$\quad z = 4 \quad ...(3)$
$\quad w = 5 \quad ...(4)$
(By definition of equality of two matrices)

Solving equations (1) and (2), we get

$x = 1, y = 2$

$\therefore \quad x + y = 1 + 2 = 3$

Q. 10. If matrix $A = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$ and $A^2 = kA$, then write the value of k. (*AI CBSE*, 2013)

Solution : $\quad A^2 = kA$

$\Rightarrow \quad AA = kA$

$$\Rightarrow \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} = k\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 1+1 & -1-1 \\ -1-1 & 1+1 \end{bmatrix} = \begin{bmatrix} k & -k \\ -k & k \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 2 & -2 \\ -2 & 2 \end{bmatrix} = \begin{bmatrix} k & -k \\ -k & 2 \end{bmatrix}$$

$\Rightarrow$ $k = 2; -k = -2$ (By definition of equality of two matrices)

$\therefore$ $k = 2$

Q. 11. If matrix $A = \begin{bmatrix} 2 & -2 \\ -2 & 2 \end{bmatrix}$ and $A^2 = pA$, then find the value of p. *(AI CBSE, 2013)*

Solution : $A^2 = pA$

$\Rightarrow$ $A \times A = P \times A$

$$\Rightarrow \begin{bmatrix} 2 & -2 \\ -2 & 2 \end{bmatrix}\begin{bmatrix} 2 & -2 \\ -2 & 2 \end{bmatrix} = p\begin{bmatrix} 2 & -2 \\ -2 & 2 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 8 & -8 \\ -8 & 8 \end{bmatrix} = \begin{bmatrix} 2p & -2p \\ -2p & 2p \end{bmatrix}$$

$\Rightarrow$ $2p = 8$

and $-2p = -8$

$\Rightarrow$ $p = 4$

Q. 12. If matrix $A = \begin{bmatrix} 3 & -3 \\ -3 & 3 \end{bmatrix}$ and $A^2 = \lambda A$, then find the value of λ. *(AI CBSE, 2013)*

Solution : $A^2 = \lambda A$

$\Rightarrow$ $A \times A = \lambda \times A$

$$\Rightarrow \begin{bmatrix} 3 & -3 \\ -3 & 3 \end{bmatrix}\begin{bmatrix} 3 & -3 \\ -3 & 3 \end{bmatrix} = \lambda\begin{bmatrix} 3 & -3 \\ -3 & 3 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 18 & -18 \\ -18 & 18 \end{bmatrix} = \begin{bmatrix} 3\lambda & -3\lambda \\ -3\lambda & 3\lambda \end{bmatrix}$$

$\Rightarrow$ $3\lambda = 18$

$-3\lambda = -18$

and $\lambda = 6$

Q. 13. Let $A = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$, show by mathematical induction that $A^n = \begin{bmatrix} \cos n\theta & \sin n\theta \\ -\sin n\theta & \cos n\theta \end{bmatrix}$, for every positive integer n.

Solution : For $n = 1$, $A^n = \begin{bmatrix} \cos n\theta & \sin n\theta \\ -\sin n\theta & \cos n\theta \end{bmatrix}$

$$\Rightarrow A^1 = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$$

or $A = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$ which is true.

$\therefore$ True for $n = 1$

Let the result be true for $n = k$,

i.e. Let $A^k = \begin{bmatrix} \cos k\theta & \sin k\theta \\ -\sin k\theta & \cos k\theta \end{bmatrix}$ ...(1)

Now $A^{k+1} = A^k.A = \begin{bmatrix} \cos k\theta & \sin k\theta \\ -\sin k\theta & \cos k\theta \end{bmatrix}\begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$ [From (1)]

$$= \begin{bmatrix} \cos k\theta\cos\theta - \cos k\theta\sin\theta & \cos k\theta\sin\theta + \sin k\theta\cos\theta \\ -\sin k\theta\cos\theta - \cos k\theta\sin\theta & -\sin k\theta\sin\theta + \cos k\theta\cos\theta \end{bmatrix}$$

$$= \begin{bmatrix} \cos(k\theta+\theta) & \sin(k\theta+\theta) \\ -\sin(k\theta+\theta) & \cos(k\theta+\theta) \end{bmatrix}$$

$$= \begin{bmatrix} \cos(k+1)\theta & \sin(k+1)\theta \\ -\sin(k+1)\theta & \cos(k+1)\theta \end{bmatrix}$$

$\Rightarrow$ The result is true for $n = k + 1$.

Therefore, the result is true for all positive integers by mathematical induction.

➠ Long Answer Type Questions

Q. 1. If $A = \begin{bmatrix} 3 & -5 \\ -4 & 2 \end{bmatrix}$ then verify that $A^2 - 5A - 14I = O$ *(BSEB, 2013)*

Solution : $A^2 = AA = \begin{bmatrix} 3 & -5 \\ -4 & 2 \end{bmatrix}\begin{bmatrix} 3 & -5 \\ -4 & 2 \end{bmatrix}$

$$= \begin{bmatrix} 9+20 & -15-10 \\ -12-8 & 20+4 \end{bmatrix}$$

$$= \begin{bmatrix} 29 & -25 \\ -20 & 24 \end{bmatrix}$$

$\therefore$ $A^2 - 5A - 14I$

$$= \begin{bmatrix} 29 & -25 \\ -20 & 24 \end{bmatrix} - 5\begin{bmatrix} 3 & -5 \\ -4 & 2 \end{bmatrix} - 14\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 29 & -25 \\ -20 & 24 \end{bmatrix} + \begin{bmatrix} -15 & 25 \\ 20 & -10 \end{bmatrix} + \begin{bmatrix} -14 & 0 \\ 0 & -14 \end{bmatrix}$$

$$= \begin{bmatrix} 29-15-14 & -25+25+0 \\ -20+20+0 & 24-10-14 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$= O$

Q. 2. Express the following matrix as the sum of a symmetric and skew-symmetric matrix and verify your result :

$$\begin{bmatrix} 3 & -2 & -4 \\ 3 & -2 & -5 \\ -1 & 1 & 2 \end{bmatrix}$$ *(AI CBSE, 2010)*

Solution : Here $A = \begin{bmatrix} 3 & -2 & -4 \\ 3 & -2 & -5 \\ -1 & 1 & 2 \end{bmatrix}$

$$\therefore \quad A' = \begin{bmatrix} 3 & 3 & -1 \\ -2 & -2 & 1 \\ -4 & -5 & 2 \end{bmatrix}$$

Now $\frac{1}{2}(A + A') + \frac{1}{2}(A - A') = P + Q$

where $P = \frac{1}{2}(A + A')$ is a symmetric and $Q = \frac{1}{2}(A - A')$ is a skew-symmetric.

$$\therefore \quad P = \frac{1}{2}\begin{bmatrix} 3+3 & -2+3 & -4-1 \\ 3-2 & -2-2 & -5+1 \\ -1-4 & 1-5 & 2+2 \end{bmatrix}$$

$$= \frac{1}{2}\begin{bmatrix} 6 & 1 & -5 \\ 1 & -4 & -4 \\ -5 & -4 & 4 \end{bmatrix} = \begin{bmatrix} 3 & 1/2 & -5/2 \\ 1/2 & -2 & -2 \\ -5/2 & -2 & 2 \end{bmatrix}$$

and $$Q = \frac{1}{2}\begin{bmatrix} 3-3 & -2-3 & -4+1 \\ 3+2 & -2+2 & -5-1 \\ -1+4 & 1+5 & 2-2 \end{bmatrix}$$

$$= \frac{1}{2}\begin{bmatrix} 0 & -5 & -3 \\ 5 & 0 & -6 \\ 3 & 6 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & -5/2 & -3/2 \\ 5/2 & 0 & -3 \\ 3/2 & 3 & 0 \end{bmatrix}$$

Now $P + Q = \begin{bmatrix} 3 & -2 & -4 \\ 3 & -2 & -5 \\ -1 & 1 & 1 \end{bmatrix}$ which is equal to A.

Q. 3. Express the matrix $A = \begin{bmatrix} 3 & 5 \\ 1 & -1 \end{bmatrix}$ as the sum of a symmetric and a skew-symmetric matrix.

(*BSEB*, 2014)

Solution : We have

$$A = \begin{bmatrix} 3 & 5 \\ 1 & -1 \end{bmatrix}$$

$$\therefore \quad A' = \begin{bmatrix} 3 & 5 \\ 1 & -1 \end{bmatrix}' = \begin{bmatrix} 3 & 1 \\ 5 & -1 \end{bmatrix}$$

Now $$A = \frac{1}{2}(A + A') + \frac{1}{2}(A - A') = P + Q$$

where $P = \frac{1}{2}(A + A')$ is symmetric and $Q = \frac{1}{2}(A - A')$ is skew-symmetric.

$$\therefore \quad P = \frac{1}{2}(A + A')$$

$$= \frac{1}{2}\left\{\begin{bmatrix} 3 & 5 \\ 1 & -1 \end{bmatrix} + \begin{bmatrix} 3 & 1 \\ 5 & -1 \end{bmatrix}\right\}$$

$$= \frac{1}{2}\begin{bmatrix} 3+3 & 5+1 \\ 1+5 & -1-1 \end{bmatrix}$$

$$= \frac{1}{2}\begin{bmatrix} 6 & 6 \\ 6 & -2 \end{bmatrix} = \begin{bmatrix} 3 & 3 \\ 3 & -1 \end{bmatrix}$$

and $$Q = \frac{1}{2}(A - A')$$

$$= \frac{1}{2}\left\{\begin{bmatrix} 3 & 5 \\ 1 & -1 \end{bmatrix} - \begin{bmatrix} 3 & 1 \\ 5 & -1 \end{bmatrix}\right\}$$

$$= \frac{1}{2}\begin{bmatrix} 3-3 & 5-1 \\ 1-5 & -1+1 \end{bmatrix}$$

$$= \frac{1}{2}\begin{bmatrix} 0 & 4 \\ -4 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 2 \\ -2 & 0 \end{bmatrix}$$

We see that

$$P + Q = \begin{bmatrix} 3 & 3 \\ 3 & -1 \end{bmatrix} + \begin{bmatrix} 0 & 2 \\ -2 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 3+0 & 3+2 \\ 3-2 & -1+0 \end{bmatrix}$$

$$= \begin{bmatrix} 3 & 5 \\ 1 & -1 \end{bmatrix} = A$$

Q. 4. If $A = \begin{bmatrix} 0 & -\tan \alpha/2 \\ \tan \alpha/2 & 0 \end{bmatrix}$ and $I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, then prove that $I + A = (I - A)\begin{bmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{bmatrix}$.

(*BSER*, 2013)

Solution : $$I + A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} 0 & -\tan\alpha/2 \\ \tan\alpha/2 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 1+0 & 0-\tan\alpha/2 \\ 0+\tan\alpha/2 & 1+0 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & -\tan\alpha/2 \\ \tan\alpha/2 & 1 \end{bmatrix} \quad \ldots(1)$$

$$I - A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} 0 & -\tan\alpha/2 \\ \tan\alpha/2 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 1-0 & 0+\tan\alpha/2 \\ 0-\tan\alpha/2 & 1-0 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & \tan\alpha/2 \\ -\tan\alpha/2 & 1 \end{bmatrix}$$

$$\therefore (I - A)\begin{bmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{bmatrix}$$

$$= \begin{bmatrix} 1 & \tan\alpha/2 \\ -\tan\alpha/2 & 1 \end{bmatrix}\begin{bmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{bmatrix}$$

$$= \begin{bmatrix} \cos\alpha + \tan\alpha/2 \sin\alpha & -\sin\alpha + \tan\alpha/2 \cos\alpha \\ -\tan\alpha\ 2\cos\alpha + \sin\alpha & \tan\alpha/2 \sin\alpha + \cos\alpha \end{bmatrix}$$

$$= \begin{bmatrix} \cos\alpha + \dfrac{\sin\frac{\alpha}{2}\sin\alpha}{\cos\frac{\alpha}{2}} & -\sin\alpha + \dfrac{\sin\frac{\alpha}{2}\cos\alpha}{\cos\frac{\alpha}{2}} \\ \dfrac{\sin\frac{\alpha}{2}\cos\alpha}{\cos\frac{\alpha}{2}} + \sin\alpha & \dfrac{\sin\frac{\alpha}{2}\sin\alpha}{\cos\frac{\alpha}{2}} + \cos\alpha \end{bmatrix}$$

$$= \begin{bmatrix} \dfrac{\cos\alpha\cos\frac{\alpha}{2} + \sin\alpha\sin\frac{\alpha}{2}}{\cos\frac{\alpha}{2}} & \dfrac{-\sin\alpha\cos\frac{\alpha}{2} + \sin\frac{\alpha}{2}\cos\alpha}{\cos\frac{\alpha}{2}} \\ \dfrac{-\sin\frac{\alpha}{2}\cos\alpha + \sin\alpha\cos\frac{\alpha}{2}}{\cos\frac{\alpha}{2}} & \dfrac{\sin\frac{\alpha}{2}\sin\alpha + \cos\alpha\cos\frac{\alpha}{2}}{\cos\frac{\alpha}{2}} \end{bmatrix}$$

$$= \begin{bmatrix} \dfrac{\cos\left(\alpha - \dfrac{\alpha}{2}\right)}{\cos\dfrac{\alpha}{2}} & \dfrac{\sin\left(\dfrac{\alpha}{2} - \alpha\right)}{\cos\dfrac{\alpha}{2}} \\ \dfrac{\sin\left(\alpha - \dfrac{\alpha}{2}\right)}{\cos\dfrac{\alpha}{2}} & \dfrac{\cos\left(\alpha - \dfrac{\alpha}{2}\right)}{\cos\dfrac{\alpha}{2}} \end{bmatrix}$$

$$= \begin{bmatrix} 1 & -\tan\alpha/2 \\ -\tan\alpha/2 & 1 \end{bmatrix} \quad \ldots(2)$$

From (1) and (2), we get

$$I + A = (I - A)\begin{bmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{bmatrix}$$

Q. 5. If $A = \begin{bmatrix} 2 & 0 & 1 \\ 2 & 1 & 3 \\ 1 & -1 & 0 \end{bmatrix}$**, then find the value of** $A^2 - 5A + 6I$**.** *(BSER, 2014)*

Solution :

$$A^2 = AA = \begin{bmatrix} 2 & 0 & 1 \\ 2 & 1 & 3 \\ 1 & -1 & 0 \end{bmatrix}\begin{bmatrix} 2 & 0 & 1 \\ 2 & 1 & 3 \\ 1 & -1 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 4+0+1 & 0+0-1 & 2+0+0 \\ 4+2+3 & 0+1-3 & 2+3+0 \\ 2-2+0 & 0-1+0 & 1-3+0 \end{bmatrix}$$

$$= \begin{bmatrix} 5 & -1 & 2 \\ 9 & -2 & 5 \\ 0 & -1 & -2 \end{bmatrix}$$

$\therefore A^2 - 5A + 6I$

$$= \begin{bmatrix} 5 & -1 & 2 \\ 9 & -2 & 5 \\ 0 & -1 & -2 \end{bmatrix} - 5\begin{bmatrix} 2 & 0 & 1 \\ 2 & 1 & 3 \\ 1 & -1 & 0 \end{bmatrix} + 6\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 5 & -1 & 2 \\ 9 & -2 & 5 \\ 0 & -1 & -2 \end{bmatrix} + \begin{bmatrix} -10 & 0 & -5 \\ -10 & -5 & -15 \\ -5 & 5 & 0 \end{bmatrix} + \begin{bmatrix} 6 & 0 & 0 \\ 0 & 6 & 0 \\ 0 & 0 & 6 \end{bmatrix}$$

$$= \begin{bmatrix} 5-10+6 & -1+0+0 & 2-5+0 \\ 9-10+0 & -2-5+6 & 5-15+0 \\ 0-5+0 & -1+5+0 & -2+0+6 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & -1 & -3 \\ -1 & -1 & -10 \\ -5 & 4 & 4 \end{bmatrix}$$

Q. 6. If $A = \begin{bmatrix} 3 & 2 \\ 7 & 5 \end{bmatrix}$ **and** $B = \begin{bmatrix} 6 & 7 \\ 8 & 9 \end{bmatrix}$**, then show that** $(AB)^{-1} = B^{-1}A^{-1}$**.** *(JAC, 2013)*

Solution : $AB = \begin{bmatrix} 3 & 2 \\ 7 & 5 \end{bmatrix}\begin{bmatrix} 6 & 7 \\ 8 & 9 \end{bmatrix}$

$$= \begin{bmatrix} 18+16 & 21+18 \\ 42+40 & 49+45 \end{bmatrix}$$

$$= \begin{bmatrix} 34 & 39 \\ 82 & 94 \end{bmatrix}$$

$$|AB| = \begin{vmatrix} 34 & 39 \\ 82 & 94 \end{vmatrix}$$

$= (34)(94) - (82)(39)$

$= 3196 - 3198$

$= -2 \neq 0$

$\therefore$ AB is invertible.

$\therefore (AB)^{-1}$ exists.

$$\text{adj}(AB) = \begin{bmatrix} 94 & -82 \\ -39 & 34 \end{bmatrix}' = \begin{bmatrix} 94 & -39 \\ -82 & 34 \end{bmatrix}$$

$$\therefore \quad (AB)^{-1} = \frac{\text{adj}(AB)}{|AB|}$$

$$= -\frac{1}{2}\begin{bmatrix} 94 & -39 \\ -82 & 34 \end{bmatrix}$$

$$= \begin{bmatrix} -47 & \dfrac{39}{2} \\ 41 & -17 \end{bmatrix} \quad \ldots(1)$$

Again $|B| = \begin{vmatrix} 6 & 7 \\ 8 & 9 \end{vmatrix} = 54 - 56 = -2 \neq 0$

$\therefore B^{-1}$ exists.

$$\text{adj}\,B = \begin{bmatrix} 9 & -8 \\ -7 & 6 \end{bmatrix}' = \begin{bmatrix} 9 & -7 \\ -8 & 6 \end{bmatrix}$$

$$\therefore \quad B^{-1} = \frac{\text{adj}\,B}{|B|} = -\frac{1}{2}\begin{bmatrix} 9 & -7 \\ -8 & 6 \end{bmatrix}$$

$$= \begin{bmatrix} -\dfrac{9}{2} & \dfrac{7}{2} \\ 4 & -3 \end{bmatrix}$$

$$|A| = \begin{vmatrix} 3 & 2 \\ 7 & 5 \end{vmatrix} = 15 - 14 = 1 \neq 0$$

$\therefore A^{-1}$ exists.

$$\text{adj}\,A = \begin{bmatrix} 5 & -7 \\ -2 & 3 \end{bmatrix}' = \begin{bmatrix} 5 & -2 \\ -7 & 3 \end{bmatrix}$$

$$\therefore \quad A^{-1} = \frac{\text{adj}\,A}{|A|} = \begin{bmatrix} 5 & -2 \\ -7 & 3 \end{bmatrix}$$

$$\therefore \quad B^{-1}A^{-1} = \begin{bmatrix} -\dfrac{9}{2} & \dfrac{7}{2} \\ 4 & -3 \end{bmatrix}\begin{bmatrix} 5 & -2 \\ -7 & 3 \end{bmatrix}$$

$$= \begin{bmatrix} -\dfrac{45}{2} - \dfrac{49}{2} & 9 + \dfrac{21}{2} \\ 20 + 21 & -8 - 9 \end{bmatrix}$$

$$= \begin{bmatrix} -47 & \dfrac{39}{2} \\ 41 & -17 \end{bmatrix} \quad \ldots(2)$$

From (1) and (2), we get

$$(AB)^{-1} = B^{-1}A^{-1}$$

Q. 7. Using elementary transformation, find the inverse of the matrix $\begin{bmatrix}3 & 10\\2 & 7\end{bmatrix}$. *(BSER, 2014)*

Solution : Let $A = \begin{bmatrix}3 & 10\\2 & 7\end{bmatrix}$

Take $A = IA$

$$\Rightarrow \begin{bmatrix}3 & 10\\2 & 7\end{bmatrix} = \begin{bmatrix}1 & 0\\0 & 1\end{bmatrix}A$$

Applying $R_1 \to R_1 - R_2$

$$\begin{bmatrix}1 & 3\\2 & 7\end{bmatrix} = \begin{bmatrix}1 & -1\\0 & 1\end{bmatrix}A$$

Applying $R_2 \to R_2 - 2R_1$

$$\begin{bmatrix}1 & 3\\0 & 1\end{bmatrix} = \begin{bmatrix}1 & -1\\-2 & 3\end{bmatrix}A$$

Applying $R_1 \to R_1 - 3R_2$

$$\begin{bmatrix}1 & 0\\0 & 1\end{bmatrix} = \begin{bmatrix}7 & -10\\-2 & 3\end{bmatrix}A$$

$$\therefore \quad A^{-1} = \begin{bmatrix}7 & -10\\-2 & 3\end{bmatrix}$$

Q. 8. If A is a square matrix such that $A^2 = A$, then write the value of $7A - (I + A)^3$, where I is an identity matrix. *(AI CBSE, 2014)*

Solution : $7A - (I + A)^3$

$$\begin{aligned}
&= 7A - (I + A)^2 (I + A)\\
&= 7A - \{(I + A)(I + A)\}(I + A)\\
&= 7A - \{I.I + IA + AI + AA\}(I + A)\\
&= 7A - (I^2 + AI + AI + A^2)(I + A)\\
&= 7A - (I + 2AI + A^2)(I + A)\\
&= 7A - (I + 2A + A^2)(I + A)\\
&= 7A - (I + 2A + A)(I + A) \quad (\because A^2 = A)\\
&= 7A - (I + 3A)(I + A)\\
&= 7A - (I.I + IA + 3AI + 3A\,A)\\
&= 7A - (I^2 + AI + 3AI + 3A^2)\\
&= 7A - (I + A + 3A + 3A) \quad (\because A^2 = A)\\
&= 7A - (I + 7A)\\
&= 7A - I - 7A\\
&= -I
\end{aligned}$$

Q. 9. Using elementary operations, find inverse of the following matrix :

$$\begin{bmatrix}-1 & 1 & 2\\1 & 2 & 3\\3 & 1 & 1\end{bmatrix}$$ *(CBSE, Delhi, 2012)*

Solution : We know that : A = IA

$$\begin{bmatrix}-1 & 1 & 2\\1 & 2 & 3\\3 & 1 & 1\end{bmatrix} = \begin{bmatrix}1 & 0 & 0\\0 & 1 & 0\\0 & 0 & 1\end{bmatrix}A$$

$$\Rightarrow \begin{bmatrix}1 & 2 & 3\\-1 & 1 & 2\\3 & 1 & 1\end{bmatrix} = \begin{bmatrix}0 & 1 & 0\\1 & 0 & 0\\0 & 0 & 1\end{bmatrix}A \text{ (applying } R_1 \leftrightarrow R_2)$$

$$\Rightarrow \begin{bmatrix}1 & 2 & 3\\0 & 3 & 5\\0 & -5 & -8\end{bmatrix} = \begin{bmatrix}0 & 1 & 0\\1 & 1 & 0\\0 & -3 & 1\end{bmatrix}A$$

(applying $R_2 \to R_2 + R_1$ and $R_3 \to R_3 - 3R_1$)

$$\Rightarrow \begin{bmatrix}1 & 2 & 3\\0 & 1 & \frac{5}{3}\\0 & -5 & -8\end{bmatrix} = \begin{bmatrix}0 & 1 & 0\\1/3 & 1/3 & 0\\0 & -3 & 1\end{bmatrix}A$$

(applying $R_2 \to \frac{1}{3}R_2$)

$$\Rightarrow \begin{bmatrix}1 & 0 & -1/3\\0 & 1 & 5/3\\0 & 0 & 1/3\end{bmatrix} = \begin{bmatrix}-\frac{2}{3} & \frac{1}{3} & 0\\\frac{1}{3} & \frac{1}{3} & 0\\\frac{5}{3} & -\frac{4}{3} & 1\end{bmatrix}A$$

$\begin{pmatrix}\text{applying } R_1 \to R_1 - 2R_2\\ \text{and } \quad R_3 \to R_3 + 5R_2\end{pmatrix}$

$$\Rightarrow \begin{bmatrix}1 & 0 & 0\\0 & 1 & 0\\0 & 0 & \frac{1}{3}\end{bmatrix} = \begin{bmatrix}1 & -1 & 1\\-8 & 7 & -5\\\frac{5}{3} & -\frac{4}{3} & 1\end{bmatrix}$$

$\begin{pmatrix}\text{applying } R_1 \to R_1 + R_3\\ \text{and } \quad R_2 \to R_2 + 5R_3\end{pmatrix}$

$$\Rightarrow \begin{bmatrix}1 & 0 & 0\\0 & 1 & 0\\0 & 0 & 1\end{bmatrix} = \begin{bmatrix}1 & -1 & 1\\-8 & 7 & -5\\5 & -4 & 3\end{bmatrix}A$$

(applying $R_3 \to 3R_3$)

$$\text{Hence,} \quad A^{-1} = \begin{bmatrix}1 & -1 & 1\\-8 & 7 & -5\\5 & -4 & 3\end{bmatrix}$$

Q. 10. Name the square matrix $A = [a_{ij}]$ in which $a_{ij} = 0, i \neq j$. *(USEB, 2015)*

Solution : $A = [a_{ij}]$ in which $a_{ij} = 0, i \neq j$

i.e. $A = \begin{bmatrix}k & 0 & 0\\0 & k & 0\\0 & 0 & k\end{bmatrix}$

This type of matrix is called "scalar matrix.

Q. 11. If $A = \begin{bmatrix}1 & 0\\1 & 0\end{bmatrix}$ and $B = \begin{bmatrix}0 & 0\\1 & 0\end{bmatrix}$ then evaluate AB. *(USEB, 2015)*

Solution : $AB = \begin{bmatrix}1 & 0\\1 & 0\end{bmatrix} \cdot \begin{bmatrix}0 & 0\\1 & 0\end{bmatrix}$

$$= \begin{bmatrix} 0+0 & 0+0 \\ 0+0 & 0+0 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

Q. 12. If $2A + B = \begin{bmatrix} 3 & -1 \\ 2 & 4 \end{bmatrix}$ and $B = \begin{bmatrix} -1 & -5 \\ 0 & 2 \end{bmatrix}$ then find A. *(Raj. Board, 2015)*

Solution : $2A + B = \begin{bmatrix} 3 & -1 \\ 2 & 4 \end{bmatrix}$

$$\Rightarrow \quad 2A + \begin{bmatrix} -1 & -5 \\ 0 & 2 \end{bmatrix} = \begin{bmatrix} 3 & -1 \\ 2 & 4 \end{bmatrix}$$

$$\Rightarrow \quad 2A = \begin{bmatrix} 3 & -1 \\ 2 & 4 \end{bmatrix} - \begin{bmatrix} -1 & -5 \\ 0 & 2 \end{bmatrix}$$

$$\Rightarrow \quad 2A = \begin{bmatrix} 3+1 & -1+5 \\ 2-0 & 4-2 \end{bmatrix} = \begin{bmatrix} 4 & 4 \\ 2 & 2 \end{bmatrix}$$

$$\Rightarrow \quad 2A = 2\begin{bmatrix} 2 & 2 \\ 1 & 1 \end{bmatrix}$$

$$\Rightarrow \quad A = \begin{bmatrix} 2 & 2 \\ 1 & 1 \end{bmatrix}$$

Q. 13. A = [1 2 3] and $B = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$, then find (AB)′. *(Raj. Board, 2015)*

Solution : $AB = [1\ 2\ 3]\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$

$$= [1 \times 1 + 2 \times 2 + 3 \times 3]$$

$$= [14]$$

then $(AB)' = [14]$

Q. 14. If $A = \begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \\ 2 & 2 & 1 \end{bmatrix}$ and $A^2 - 4A = kI_3$, then find the value of k. *(Raj. Board, 2015)*

Solution : $A = \begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \\ 2 & 2 & 1 \end{bmatrix}$

$$\therefore \quad A^2 = A . A = \begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \\ 2 & 2 & 1 \end{bmatrix}\begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \\ 2 & 2 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 9 & 8 & 8 \\ 8 & 9 & 8 \\ 8 & 8 & 9 \end{bmatrix}$$

$$\therefore \quad A^2 - 4A = \begin{bmatrix} 9 & 8 & 8 \\ 8 & 9 & 8 \\ 8 & 8 & 9 \end{bmatrix} - 4\begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \\ 2 & 2 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 9 & 8 & 8 \\ 8 & 9 & 8 \\ 8 & 8 & 9 \end{bmatrix} - \begin{bmatrix} 4 & 8 & 8 \\ 8 & 4 & 8 \\ 8 & 8 & 4 \end{bmatrix}$$

$$= \begin{bmatrix} 5 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 5 \end{bmatrix}$$

$$= 5\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$A^2 - 4A = 5I_3$ ...(1)

and given that $A^2 - 4A = kI_3$ ...(3)

comparing equations (1) and (2)

$k = 5$

Q. 15. Construct a 3 × 2 matrix whose $(i, j)^{th}$ element $a_{ij} = \frac{1}{2}\,|i - 3j|$. *(JAC, 2015)*

Solution : $\because \quad a_{ij} = \frac{1}{2}\,|i - 3j|$

$$\therefore \quad a_{11} = \frac{1}{2}|1-3| = \frac{1}{2} \times 2 = 1$$

$$a_{12} = \frac{1}{2}|1-6| = \frac{5}{2}$$

$$a_{21} = \frac{1}{2}|2-3| = \frac{1}{2}$$

$$a_{22} = \frac{1}{2}|2-6| = \frac{1}{2} \times 4 = 2$$

$$a_{31} = \frac{1}{2}|3-3| = 0$$

$$a_{32} = \frac{1}{2}|3-6| = \frac{3}{2}$$

thus, required matrix $= \begin{bmatrix} 1 & 5/2 \\ 1/2 & 2 \\ 0 & 3/2 \end{bmatrix}$

Q. 16. If $A = \begin{bmatrix} 2 & -1 \\ 0 & 3 \end{bmatrix}$ then find A^2. *(JAC, 2015)*

Solution : $A = \begin{bmatrix} 2 & -1 \\ 0 & 3 \end{bmatrix}$

$$\therefore \quad A^2 = A . A = \begin{bmatrix} 2 & -1 \\ 0 & 3 \end{bmatrix}\begin{bmatrix} 2 & -1 \\ 0 & 3 \end{bmatrix}$$

$$= \begin{bmatrix} 4+0 & -2-3 \\ 0+0 & 0+9 \end{bmatrix}$$

$$= \begin{bmatrix} 4 & -5 \\ 0 & 9 \end{bmatrix}$$

Q. 17. Find the values of x and y : *(BSEB, 2015)*

$$x + y = \begin{bmatrix} 7 & 0 \\ 2 & 5 \end{bmatrix} \text{ and } x - y = \begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix}$$

Solution : $\because$ $x + y = \begin{bmatrix} 7 & 0 \\ 2 & 5 \end{bmatrix}$ ...(1)

and $x - y = \begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix}$ ...(2)

Adding equations (1) and (2), we get

$$2x = \begin{bmatrix} 7+3 & 0+0 \\ 2+0 & 5+3 \end{bmatrix} = \begin{bmatrix} 10 & 0 \\ 2 & 8 \end{bmatrix}$$

$$\Rightarrow \quad 2x = 2\begin{bmatrix} 5 & 0 \\ 1 & 4 \end{bmatrix}$$

$$\Rightarrow \quad x = \begin{bmatrix} 5 & 0 \\ 1 & 4 \end{bmatrix}$$

Again subtracting equation (2) from equation (1),

$$2y = \begin{bmatrix} 7-3 & 0-0 \\ 2-0 & 5-3 \end{bmatrix} = \begin{bmatrix} 4 & 0 \\ 2 & 2 \end{bmatrix}$$

$$\Rightarrow \quad 2y = 2\begin{bmatrix} 2 & 0 \\ 1 & 1 \end{bmatrix}$$

$$\Rightarrow \quad y = \begin{bmatrix} 2 & 0 \\ 1 & 1 \end{bmatrix}$$

Q. 18. With the help of elementary operations, find the inverse of the following matrix :

$$\begin{bmatrix} 2 & 0 & -1 \\ 5 & 1 & 0 \\ 0 & 1 & 3 \end{bmatrix}$$ (*USEB*, 2015)

Solution : Let $A = \begin{bmatrix} 2 & 0 & -1 \\ 5 & 1 & 0 \\ 0 & 1 & 3 \end{bmatrix}$

$\because$ $A = IA$

$$\Rightarrow \quad \begin{bmatrix} 2 & 0 & -1 \\ 5 & 1 & 0 \\ 0 & 1 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} A$$

Applying $R_1 \to 3R_1$

$$\begin{bmatrix} 6 & 0 & -3 \\ 5 & 1 & 0 \\ 0 & 1 & 3 \end{bmatrix} = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} A$$

Applying $R_1 \to R_1 - R_2$

$$\begin{bmatrix} 1 & -1 & -3 \\ 5 & 1 & 0 \\ 0 & 1 & 3 \end{bmatrix} = \begin{bmatrix} 3 & -1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} A$$

Applying $R_1 \to R_1 + R_3$

$$\begin{bmatrix} 1 & 0 & 0 \\ 5 & 1 & 0 \\ 0 & 1 & 3 \end{bmatrix} = \begin{bmatrix} 3 & -1 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} A$$

Applying $R_2 \to R_2 - 5R_1$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 3 \end{bmatrix} = \begin{bmatrix} 3 & -1 & 1 \\ -15 & 6 & -5 \\ 0 & 0 & 1 \end{bmatrix} A$$

Applying $R_3 \to R_3 - R_2$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 3 \end{bmatrix} = \begin{bmatrix} 3 & -1 & 1 \\ -15 & 6 & -5 \\ 15 & -6 & 6 \end{bmatrix} A$$

Applying $R_3 \to \frac{1}{3} R_3$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 3 & -1 & 1 \\ -15 & 6 & -5 \\ 5 & -2 & 2 \end{bmatrix} A$$

Thus $A^{-1} = \begin{bmatrix} 3 & -1 & 1 \\ -15 & 6 & -5 \\ 5 & -2 & 2 \end{bmatrix}$

NCERT QUESTIONS

Q. 1. Using elementary transformations, find the inverse of the matrix :

$$\begin{bmatrix} 1 & 3 & -2 \\ -3 & 0 & -1 \\ 2 & 1 & 0 \end{bmatrix}$$

(*CBSE*, 2011; JAC, 2014)

Solution : We have

$$A = IA$$

$$\Rightarrow \quad \begin{bmatrix} 1 & 3 & -2 \\ -3 & 0 & -1 \\ 2 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} A$$

Applying $R_2 \to R_2 + 3R_1$ and $R_3 \to R_3 - 2R_1$

$$\begin{bmatrix} 1 & 3 & -2 \\ 0 & 9 & -7 \\ 0 & -5 & 4 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ -2 & 0 & 1 \end{bmatrix} A$$

Applying $R_2 \to \frac{1}{9} R$

$$\begin{bmatrix} 1 & 3 & -2 \\ 0 & 1 & -\frac{7}{9} \\ 0 & -5 & 4 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ \frac{1}{3} & \frac{1}{9} & 0 \\ -2 & 0 & 1 \end{bmatrix} A$$

Applying $R_1 \to R_1 - 3R_2$ and $R_3 \to R_3 + 5R_2$

$$\begin{bmatrix} 1 & 0 & \frac{1}{3} \\ 0 & 1 & -\frac{7}{9} \\ 0 & 0 & \frac{1}{9} \end{bmatrix} = \begin{bmatrix} 0 & -\frac{1}{3} & 0 \\ \frac{1}{3} & \frac{1}{9} & 0 \\ -\frac{1}{3} & \frac{5}{9} & 1 \end{bmatrix} A$$

Applying $R_3 \to 9R_3$

$$\begin{bmatrix} 1 & 0 & \frac{1}{3} \\ 0 & 1 & -\frac{7}{9} \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & -\frac{1}{3} & 0 \\ \frac{1}{3} & \frac{1}{9} & 0 \\ -3 & 5 & 9 \end{bmatrix} A$$

Applying $R_1 \to R_1 - \frac{R_3}{3}$ and $R_2 \to R_2 + \frac{7}{9} R^3$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & -2 & -3 \\ -2 & 4 & 7 \\ -3 & 5 & 9 \end{bmatrix} A$$

Hence $\quad A^{-1} = \begin{bmatrix} 1 & -2 & -3 \\ -2 & 4 & 7 \\ -3 & 5 & 9 \end{bmatrix}$

Q. 2. Find X and Y if :

(i) $X + Y = \begin{bmatrix} 7 & 0 \\ 2 & 5 \end{bmatrix}$ **and** $X - Y = \begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix}$

(ii) $2X + 3Y = \begin{bmatrix} 2 & 3 \\ 4 & 0 \end{bmatrix}$ **and** $3X + 2Y = \begin{bmatrix} 2 & -2 \\ -1 & 5 \end{bmatrix}$.

(*USEB*, 2014)

Solution.

(i) $\quad X + Y = \begin{bmatrix} 7 & 0 \\ 2 & 5 \end{bmatrix} \quad \ldots(1)$

$X - Y = \begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix} \quad \ldots(2)$

Adding equations (1) and (2),

$$2X = \begin{bmatrix} 10 & 0 \\ 2 & 8 \end{bmatrix}$$

$$\Rightarrow \quad X = \begin{bmatrix} 5 & 0 \\ 1 & 4 \end{bmatrix}$$

Subtracting equation (2) from equation (1),

$$2Y = \begin{bmatrix} 4 & 0 \\ 2 & 2 \end{bmatrix} \Rightarrow Y = \begin{bmatrix} 2 & 0 \\ 1 & 1 \end{bmatrix}$$

(ii) $\quad 2X + 3Y = \begin{bmatrix} 2 & 3 \\ 4 & 0 \end{bmatrix} \quad \ldots(1)$

$3X + 2Y = \begin{bmatrix} 2 & -2 \\ -1 & 5 \end{bmatrix} \quad \ldots(2)$

(1) × 3 – (2) × 2,

$$9Y - 4Y = \begin{bmatrix} 6 & 9 \\ 12 & 0 \end{bmatrix} - \begin{bmatrix} 4 & -4 \\ -2 & 10 \end{bmatrix}$$

$$\Rightarrow \quad 5Y = \begin{bmatrix} 2 & 13 \\ 14 & -10 \end{bmatrix}$$

$$\therefore \quad Y = \frac{1}{5}\begin{bmatrix} 2 & 13 \\ 14 & -10 \end{bmatrix} = \begin{bmatrix} \frac{2}{5} & \frac{13}{5} \\ \frac{14}{5} & -2 \end{bmatrix}$$

Now (1) × 2 – (2) × 3,

$$4X - 9X = \begin{bmatrix} 4 & 6 \\ 8 & 0 \end{bmatrix} - \begin{bmatrix} 6 & -6 \\ -3 & 15 \end{bmatrix}$$

$$\Rightarrow \quad -5X = \begin{bmatrix} -2 & 12 \\ 11 & -15 \end{bmatrix}$$

$$\Rightarrow \quad X = \frac{1}{5}\begin{bmatrix} -2 & 12 \\ 11 & -15 \end{bmatrix} = \begin{bmatrix} \frac{2}{5} & -\frac{12}{5} \\ -\frac{11}{5} & 3 \end{bmatrix}$$

Q. 3. If $A = \begin{bmatrix} 3 & 1 \\ -1 & 2 \end{bmatrix}$ **then show that** $A^2 - 5A + 7I = O$**, use this result to find** A^4 **and** A^{-1}**.** (*BSER*, 2013)

Solution : $\quad A = \begin{bmatrix} 3 & 1 \\ -1 & 2 \end{bmatrix}$

$$A^2 = \begin{bmatrix} 3 & 1 \\ -1 & 2 \end{bmatrix}\begin{bmatrix} 3 & 1 \\ -1 & 2 \end{bmatrix} = \begin{bmatrix} 8 & 5 \\ -5 & 3 \end{bmatrix}$$

$$A^3 = A^2 \times A = \begin{bmatrix} 8 & 5 \\ -5 & 3 \end{bmatrix}\begin{bmatrix} 3 & 1 \\ -1 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} 19 & 18 \\ -18 & 1 \end{bmatrix}$$

Now $A^2 - 5A + 7I = \begin{bmatrix} 8 & 5 \\ -5 & 3 \end{bmatrix} - 5\begin{bmatrix} 3 & 1 \\ -1 & 2 \end{bmatrix} + 7\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

$$= \begin{bmatrix} 8 & 5 \\ -5 & 3 \end{bmatrix} - \begin{bmatrix} 15 & 5 \\ -5 & 10 \end{bmatrix} + \begin{bmatrix} 7 & 0 \\ 0 & 7 \end{bmatrix}$$

$$= \begin{bmatrix} 8 - 15 + 7 & 5 - 5 + 0 \\ -5 + 5 + 0 & 3 - 10 + 7 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = O$$

Multiplying by A^2, we get

$$A^4 - 5A^3 + 7I.A^2 = O$$

or $\quad A^4 = 5A^3 - 7A^2$

or $\quad A^4 = 5\begin{bmatrix} 19 & 18 \\ -18 & 1 \end{bmatrix} - 7\begin{bmatrix} 8 & 5 \\ -5 & 3 \end{bmatrix}$

$$= \begin{bmatrix} 95 & 90 \\ -90 & 5 \end{bmatrix} - \begin{bmatrix} 56 & 35 \\ -35 & 21 \end{bmatrix}$$

$$= \begin{bmatrix} 39 & 55 \\ -55 & -16 \end{bmatrix}.$$

we have

$$A^2 - 5A + 7I = O$$

Multiplying by A^{-1}, we get

$$A^{-1}A^2 - 5A^{-1}A + 7A^{-1}I = A^{-1}O$$

$$\Rightarrow \quad (A^{-1}A)A - 5I + 7A^{-1} = O$$

$$\Rightarrow \quad IA - 5I + 7A^{-1} = O$$

$$\Rightarrow \quad A - 5I + 7A^{-1} = O$$

$$\Rightarrow \quad 7A^{-1} = 5I - A$$

$$= 5\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} 3 & 1 \\ -1 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} 5 & 0 \\ 0 & 5 \end{bmatrix} - \begin{bmatrix} 3 & 1 \\ -1 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} 5-3 & 0-1 \\ 0+1 & 5-2 \end{bmatrix} = \begin{bmatrix} 2 & -1 \\ 1 & 3 \end{bmatrix}$$

$$\Rightarrow \quad A^{-1} = \frac{1}{7}\begin{bmatrix} 2 & -1 \\ 1 & 3 \end{bmatrix}$$

Q. 4. Matrix $A = \begin{bmatrix} 1 & 3 & -2 \\ -3 & 0 & -5 \\ 2 & 5 & 0 \end{bmatrix}$**, find inverse.**

Solution :

We know that $\quad A = IA$

$$\begin{bmatrix} 1 & 3 & -2 \\ -3 & 0 & -5 \\ 2 & 5 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} A$$

$$\Rightarrow \begin{bmatrix} 1 & 3 & -2 \\ 0 & 9 & -11 \\ 0 & -1 & 4 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ -2 & 0 & 1 \end{bmatrix} A$$

(applying $R_2 \to R_2 + 3R_1$, $R_3 \to R_3 - 2R_1$)

$$\Rightarrow \begin{bmatrix} 1 & 3 & -2 \\ 0 & 1 & 21 \\ 0 & -1 & 4 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -13 & 1 & 8 \\ -2 & 0 & 1 \end{bmatrix} A$$

(applying $R_2 \to R_2 + 8R_3$)

$$\Rightarrow \begin{bmatrix} 1 & 0 & 10 \\ 0 & 1 & 21 \\ 0 & -1 & 4 \end{bmatrix} = \begin{bmatrix} -5 & 0 & 3 \\ -13 & 1 & 8 \\ -2 & 0 & 1 \end{bmatrix} A$$

(applying $R_1 \to R_1 + 3R_3$)

$$\Rightarrow \begin{bmatrix} 1 & 0 & 10 \\ 0 & 1 & 21 \\ 0 & 0 & 25 \end{bmatrix} = \begin{bmatrix} -5 & 0 & 3 \\ -13 & 1 & 8 \\ -15 & 1 & 9 \end{bmatrix} A$$

(applying $R_3 \to R_3 + R_2$)

$$\Rightarrow \begin{bmatrix} 1 & 0 & 10 \\ 0 & 1 & 21 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} -5 & 0 & 3 \\ -13 & 1 & 8 \\ -\frac{3}{5} & \frac{1}{25} & \frac{9}{25} \end{bmatrix} A$$

$\left(\text{applying } R_3 \to \frac{R_3}{25}\right)$

$$\Rightarrow \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & -\frac{2}{5} & -\frac{3}{5} \\ -\frac{2}{5} & \frac{2}{25} & \frac{11}{25} \\ -\frac{3}{5} & \frac{1}{25} & \frac{9}{25} \end{bmatrix} A$$

$\left(\begin{array}{l}\text{applying } R_1 \to R_1 - 10R_3 \\ \text{and} \qquad R_2 \to R_2 - 21R_3\end{array}\right)$

$$\therefore \quad A^{-1} = \begin{bmatrix} 1 & -\frac{2}{5} & -\frac{3}{5} \\ -\frac{2}{5} & \frac{4}{25} & \frac{11}{25} \\ -\frac{3}{5} & \frac{1}{25} & \frac{9}{25} \end{bmatrix}$$

Q. 5. If $A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$**, then prove that**

$$A^n = \begin{bmatrix} 3^{n-1} & 3^{n-1} & 3^{n-1} \\ 3^{n-1} & 3^{n-1} & 3^{n-1} \\ 3^{n-1} & 3^{n-1} & 3^{n-1} \end{bmatrix}, n \in N$$

Solution : We shall prove it by mathematical induction method.

When $n = 1$

$$\text{LHS} = A' = \begin{bmatrix} 3^0 & 3^0 & 3^0 \\ 3^0 & 3^0 & 3^0 \\ 3^0 & 3^0 & 3^0 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} = \text{RHS}$$

$\therefore$ This is true for $n = 1$.

Let it is true for $n = m$, then

$$A^m = \begin{bmatrix} 3^{m-1} & 3^{m-1} & 3^{m-1} \\ 3^{m-1} & 3^{m-1} & 3^{m-1} \\ 3^{m-1} & 3^{m-1} & 3^{m-1} \end{bmatrix} \quad \ldots(1)$$

$\therefore A^{m+1} = A^m A$

$$= \begin{bmatrix} 3^{m-1} & 3^{m-1} & 3^{m-1} \\ 3^{m-1} & 3^{m-1} & 3^{m-1} \\ 3^{m-1} & 3^{m-1} & 3^{m-1} \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} \text{[Using (1)]}$$

$$= \begin{bmatrix} 3.3^{m-1} & 3.3^{m-1} & 3.3^{m-1} \\ 3.3^{m-1} & 3.3^{m-1} & 3.3^{m-1} \\ 3.3^{m-1} & 3.3^{m-1} & 3.3^{m-1} \end{bmatrix}$$

$$A^{m+1} = \begin{bmatrix} 3^{(m+1)-1} & 3^{(m+1)-1} & 3^{(m+1)-1} \\ 3^{(m+1)-1} & 3^{(m+1)-1} & 3^{(m+1)-1} \\ 3^{(m+1)-1} & 3^{(m+1)-1} & 3^{(m+1)-1} \end{bmatrix}$$

$\therefore$ It is true for $n = m + 1$.

Hence, it is true for all $n \in N$ by mathematical induction.

■■