MATHEMATICS

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AIM POINT
MATHEMATICS
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PERMUTATIONS AND COMBINATIONS

& Their Properties

CONTENTS

THINGS TO REMEMBER

✵ <u>Factorial Notation</u>

The product of first n natural numbers is deoted by n ! and read 'factorial n'.

Thus, $n! = n(n-1)(n-2) ....3\cdot2\cdot1$

eg, $5! = 5 \times 4 \times 3 \times 2 \times 1 = 120$

and $4! = 4 \times 3! = 4 \times 3 \times 2! = 4 \times 3 \times 2 \times 1 = 24$

Properties of Factorial Notation

(i) $0! = 1! = 1$

(ii) Factorials of negative integers nd fraction s are not defined.

(iii) $n! = n(n-1)! = n(n-1)(n-2)!$

(iv) $\frac{n!}{r!} = n(n-1)(n-2) ...... (r+1)$

✵ <u>Exponent fo Prime p in n !</u>

Let n be a positive integer and p be a prime number. Then, last integer amongst 1, 2, 3,......,(n – 1), n which is divisible by p is $\left[\frac{n}{p}\right]$p, where $\left[\frac{n}{p}\right]$ denoted the greatest integer less then or equal to $\frac{n}{p}$.

eg, $\left[\frac{12}{5}\right] = 2, \left[\frac{15}{5}\right] = 3$ etc

Let $E_p(n!)$ denotes the exponent of prime p in n!, then

$$E_p(n!) = \left[\frac{n}{p}\right] + \left[\frac{n}{p^2}\right] + ... + \left[\frac{n}{p^a}\right]$$

where a is a greatest positive integer such tat $p^a \le n \le p^{a+1}$.

✵ <u>Fundamental Principles of Counting</u>

1. Fundamental Principle of multiplication

If an operation can be performed in m different ways, following which second operation can be performed in n deferent ways, then the two operations in succession can be performed in m x n ways. This can be extended to any finite number of operations.

eg, A hall has 12 gets, After entering into the hall the man come out through a different gate in 11 ways.

Hence, by the fundamental principle of multiplication, the total number of ways of man come out through different gates = 12 x 11 = 132.

2. Fundamental Principle of Addition

If an operation can be performed in m different ways and another operation, which is independent of the first operation can be performed in n different ways. Then, either of the two operations can be performed in $(m + n)$ ways. This can be extended to any finite number of mutually exclusive operations.

eg, There are 25 students in a class in which 15 boys and 10 girls. The class teracher select either a boy or a girl for monitor f the class. Since there are 15 ways to select a boy and there are 10 ways to select a girl. Hence, by the fundamental principle of addition, the number of ways in which either a boys or a girl can be chosen as a monitor = 10 + 15 = 25 ways.

❋ Permutation

Each of different arrangements which can be made by taking some or all of a number of things is called a permutation.

eg, Arrangements of objects taking 2 at a time from given 3 objects (a, b, c) are ab, bc, ca, cb, ac, ba, then total number of arrangements is 6 each of which is known as permuatation.

Important Results Related to Permutations

(i) Number of permutations of n distinct objects taking r at a time is denoted by ${}^nP_r$.

$$ {}^nP_r = \frac{n!}{(n-r)!}, \forall\, 0 \le r \le n $$

$$ = n(n-1)(n-2)......(n-r+1), \ \forall\, n \in N \text{ and } r \in W. $$

(ii) The number of permutations of n things taken all at a time p are alike of one kind, q are alike of second kind, r are alike of third kind and remaining are distinct, is $\frac{n!}{p!q!r!}$.

(iii) The number of permutations of n different things, taken r at a time when each thing may be repeated any number of times is n^r.

(iv) Number of permutations under certain conditions

(a) Number of permutations of n different things taken r at a time when a particular thing is to be always included in each arrangement is $r.\,{}^{n-1}P_{r-1}$.

(b) Number of permutations of n different things taken r at a time, when a particular thing is never taken in each arrangement is ${}^{n-1}P_r$.

(c) Number of permutations of n different things taken all at a time, when m specified things always come together is $m! \times (n-m+1)!$.

(c) Number of permutations of n different things taken all at a time, when m specified things never come together is $n! - m! \times (n-m+1)!$.

❋ Circular Permutation

If objects are arranged along a closed curve, then permutation is known as circular permuatation.

Important Results Related to Circular Permutation.

(i) Number of circular permutations of n different things taken al at a time = $(n-1)!$. If clockwise and anti-clockwise orders are taken as different.

(ii) The number of circular permutations of n different thing taken all at a time = $\frac{1}{2}(n-1)!$ If clockwise and anticlockwise orders are taken as not different.

✵ Combination

Each of the different groups or selections which can be made by some or all of a number of given things without reference to the order of the things is each group is called a combination.

eg, The groups make by taking 2 objects at a time from three objects (a,b,c) are ab, bc, ca, Then, the number of groups is 3 each of which is known as combination.

Important Results Related to Combinations

(i) The number of combinations of n different things taken r at a time is denoted by ^nC_r or C(n, r) or $\binom{n}{r}$.

Then, $^nC_r = \dfrac{n!}{r!(n-r)!} = \dfrac{^nP_r}{r!}$ (0 ≤ r ≤ n)

$$= \frac{n(n-1)(n-2).......(n-r+1)}{r(r-1)(r-2)......2\cdot 1}$$

n∈ N and r ∈ W

If r > n, then $^nC_r = 0$

Properties of ^nC_r.

(a) ^nC_r is a natural number

(b) $^nC_0 = {^nC_n} = 1$, $^nC_1 = n$

(c) $^nC_r = {^nC_{n-r}}$

(d) $^nC_r + {^nC_{n-r}} = {^{n+1}C_r}$

(e) $^nC_x = {^nC_y} \Leftrightarrow x = y$ or $x + y = n$

(f) $n.{^{n-1}C_{r-1}} = (n-r+1)\ {^nC_{r-1}}$

(g) If n is even, then the greatest value of ^nC_r is $^nC_{n/2}$.

(h) If n is odd, then the greatest value of ^nC_r is $^nC_{\frac{n+1}{2}}$ or $^nC_{\frac{n-1}{2}}$.

(i) $^nC_r = \dfrac{n}{r}.\ {^{n-1}C_{r-1}}$

(j) $\dfrac{^nC_r}{^nC_{r-1}} = \dfrac{n-r+1}{r}$

(ii) The number of combinations of n different things, taken r at a time, where p particular things occur is $^{n-p}C_{r-p}$.

(iii) The number of combinations of n different things, taken r at a time, where p particular things never occur is $^{n-p}C_r$.

(iv) The total number of combinations of n different things taken one or more at a time or the number of ways of n different things selecting at least one of them is

$$^nC_r + {}^nC_r + \ldots\ldots.. + {}^nC_n = 2^n - 1$$

(v) The number of combinations of n identical things taking r ($r \le n$) at a time is 1.

(vi) The number of ways of selecting r things out of n alike things is (n + 1). (where r = 0,1,2,3,......,n).

(vii) If out of (p + q + r) things, p are alike of one kind, q are alike of second kind and rest are alike of third kind, then the total number of combinations is

$$[(p + 1)(q + 1)(r + 1)] - 1$$

(viii) If out of (p + q + r + t) things, p are alike of one kind, q are like of second kind, r alike of third kind and t are different, then he total number of combinations is $(p + 1)(q + 1)(r + 1)2^t - 1$.

(ix) Division into Group

(a) The number of ways in which (m + n) different things can be divied into two groups which contain m and n things respectively is $\frac{(m+n)!}{m!n!}$, $m \ne n$ and if m = n, then groups are of equal size. Division of these groups can be given by two types.

If order of groups is not important The number of ways in which 2n different things can be divided equally into two groups is $\frac{2n!}{2!(n!)^2}$.

If order of groups is important The number of ways in which 2n different things can be divided equally into two different groups is $\frac{2n!}{(n!)^2}$.

(b) The number of ways in which (m + n +p) different things can be divided into three groups which contain m, n and p things respectively is $\frac{(m+n+p)!}{m!n!p!}$, $m \ne n \ne p$ and m = n = p, then the groups of equal size. Division of these groups can be given by two types.

If Order of group is not Important The number of ways in which 3p different things can be divided equally into three groups is $\frac{3p!}{3!(p!)^3}$.

If order of groups is important The number of ways in which 3p different things can be divided equally into three distinct groups is $\frac{3p!}{(p!)^3}$.

(x) Arrangement in Groups

(a) The number of ways in which n different things can be arranged into r different groups is $^{n+r-1}P_n$ or $n!\ {}^{n-1}C_{r-1}$.

(b) The number of ways in which n different things can be distributed into r different groups is $r^n - {}^rC_1(r-1)^n + {}^rC_2(r-2)^n - \ldots\ldots.. + (-1)^{r-1}\ .\ {}^rC_{r-1}$. or coefficient of xn in $n!\ (e^x - 1)^r$.

Here, blank groups are not allowed.

(c) The number of ways in which n identical things can be distributed into r different groups in ${}^{n+r-1}C_{r-1}$ or ${}^{n-1}C_{r-1}$, according as blank groups are or are not admissible.

(d) The number of ways in which n identical items can be divided into r groups so that no group cotains less than m items and more than k (m < k) is coefficient of xn in the expansion of $(x^m + x^{m+1} + ...+ x^k)^r$.

Geometrical Application of Combination

(i) Out of n non-concurrent and non-parellel straight lines the number of point of intersection are ${}^nC_2$.

(ii) The number of straight line passing through n points = ${}^nC_2$.

(iii) The number of straight line passing through n points out of which m are collinear = ${}^nC_2 - {}^mC_2 + 1$.

(iv) In a polygon, the total number of diagonals out of n points (no three points are collinear) = ${}^nC_2 - n$ $= \frac{n(n-3)}{2}$.

(v) Number of triangles formed by joining n points is ${}^nC_3$.

(vi) Number of triangles formed by joining n points out of which m are collinear, are ${}^nC_3 - {}^nC_3$.

(vii) The number of parallelogram in two system of parallel lines (when Ist set contains m parallel lines and 2nd set contains n parallel lines) = ${}^nC_2 \times {}^mC_2$.

(viii) If in any party n persons are present, then total number of hand shakes = ${}^nC_2$.

❋ **Number of Divisors**

Let N = $p_1^{\alpha_1} . p_2^{\alpha_2} . p_3^{\alpha_3}$ .... $p_k^{\alpha_k}$, where $p_1, p_2, p_3, ...,p_k$ are different primes and $\alpha_1, \alpha_2, \alpha_3, ...\alpha_k$ are natural numbers, then

(i) The total number of divisors of N including 1 and N = $(\alpha_1 + 1)(\alpha_2 + 1)(\alpha_3 + 1)........(\alpha_k + 1)$.

(ii) The total number of divisors of N excluding 1 and n = $(\alpha_1 + 1)(\alpha_2 + 1)(\alpha_3 + 1)........(\alpha_k + 1) - 2$.

(iii) The total number of divisors of N excluding 1 and n = $(\alpha_1 + 1) + (\alpha_2 + 1)(\alpha_3 + 1)........(\alpha_k + 1) - 1$.

(iv) The sum of these divisors =

$(p_1^0 + p_1^1 + p_1^2 + ... + p_1^{\alpha_1})(p_1^0 + p_1^1 + p_1^2 + ... + p_2^{\alpha_2})......(p_k^0 + p_k^1 + p_k^2 + ...+ p_k^{\alpha_k})$
(Use sum of GP in each bracket)

(v) The number of ways in which N can be resolved as a product of two factors

$\frac{1}{2}(\alpha_1 + 1)(\alpha_2 + 1)........(\alpha_k + 1)$ is if N is not a perfect square and $\frac{1}{2}[(\alpha_1 + 1)(\alpha_2 + 1)....(\alpha_k + 1) + 1]$, if N is a perfect square.

(vi) The number of ways in which a composite number N can be resolved into two factors which are relatively prime (or coprime) to each other is equal 2^{n-1}, where n is the number of different factors in N.

❋ Dearrangements

If n district objects are arranged in a row, then the number of ways in which they can be dearranged so that none of them occupies its original place is

$$n!\left\{1-\frac{1}{1!}+\frac{1}{2!}-\frac{1}{3!}+\frac{1}{4!}-.....+(-1)^n\frac{1}{n!}\right\}$$

and it is denoted by D(n).

If r ($0 \le r \le n$) objects occupy the places assined to them ie, their originalplace and none of the remaining (n – r) objects occupies its original plsces then the number of such ways is D (n – r) = ${}^nC_r$. D(n – r)

$$= {}^nC_r \,.\, (n-r)!\ \left\{1-\frac{1}{1!}+\frac{1}{2!}-\frac{1}{3!}+\frac{1}{4!}-.....+(-1)^n\frac{1}{n!}\right\}.$$

Note :

- The number of permutations of n distinct object taken all at a time is ${}^nP_n = n!$
- ${}^nP_0 = 1$, ${}^nP_1 = n$ and ${}^nP_{n-1} = n\,!$
- ${}^nP_0 = n$, ${}^{n-1}P_{r-1} = r\,.\,{}^{n-1}P_{r-1} + {}^{n-1}P_r$
- ${}^{n-1}P_r = (n-r){}^{n-1}P_{r-1}$
- Number of permutations of n different things taken r at a time when p particular things are to be always included in each arrangement is $p!\,(r-(p-1)){}^{n-p}P_{r-p}$.
- In permutation order of object is improtant whereas is combination order of objects is not important.