MATHEMATICS

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MATHEMATICS
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PROBABILITY

& Their Properties

CONTENTS

THINGS TO REMEMBER

✷ **Basic Terminology**

1. **Rendom Experiment**

An operation which results in some will defined outcomes is called an experiment.

An experiment in which total outcomes are known in advance but occurrence of specific outcome can be told only after completion of the experiment is known as a random experiment.

OR

If in each trial of an experiment, which when repeated under indentical conditions, the outcome is not unique but the outcome in a trial is one of the several possible outcomes, then such an experiment is known as a rendom experiment.

eg, Tossing a coin, throwing a dice, selecting a card from a pack of playing cards etc. In all these cases there are a number of possible results which can occur but there is an uncertainty as to which one of them will actually occur.

2. **Sample Space**

The set of total possible outcomes in a random experiment is known a sample sample space for that experiment. Each element of te sample space is called a spample point or an event point. Generally sample space in denoted by S.

A sammple space is called descrete sample space if it is finite e, In tossing of two coins, sample space is {HH, HT, TH, TT} HH, HT, TH, TT are sample points.

3. **Trial and Event**

Any particular ferformance of a random experiment is called a trial and outcome or combination of outcomes are termed as events.

eg, If a coin is tossed repeatedly, the result is not unique. We may get any of the two faces head or tail.

Thus, tossing of a coin is a random experiment or trail and getting of a head or tail is an event.

Event is called simple if it correcponds to a single possible outcome of the experiment otherwise it is known as a compound or composite event.

eg, In tossing of a single die the event of getting '6' is a simple event but the event of getting an even number is a composite event.

4. **Equally Likely Events**

The given events are said to be equally likely, if none of them is expected to occur in perference to the other.

eg, In throwing an unbiased die, all the six faces are equally likely to come.

5. **Mutually Exclusive Events**

Events are said to be mutually exclusive or incompatible if the happening of any one of them precludes the heppening of all the others ie, if no two or more ot them can happen simltaneously in the same trial.

OR

Two or more events associted to a random experiment are said to be murually exclusive events, if the occurrence of one of them pervents or denies the occurrence of all others.

eg, In throwing a die all the 6 faces numbere 1 to 6 are mutually exclusive since if any one of these faces comes, the possibility of thers, in the same trial, is ruled out.

6. Exhustive Events

The total number of possible outcomes of a random experiment is known as the exhustive events or cases.

OR

Two or more events associated to a random experiments are exhustive, if their union is the sample space.

ie, Events $A_1, A_2, A_3, ..., A_n$ associated to a random experiment with sample space are exhustive, if.

$$A_1 \cup A_2 \cup .... \cup A_n = S.$$

eg, In throwing of two dice, the exhaustive number of cases is $6^2 = 36$, since any of the number 1 to 6 on the first die can be accociated with any of the 6 numbers on the other die.

7. Favourble Events

The number of cases favourable to an event in a trial is the number of cutocomes which entail the happening of the even.

eg, In drawing a card from a pack of cards the number of cases favourable to drawing of n ace is 4, for drawing a spade is 13 and for drawing a red card is 26.

8. Independent Events

Two events A and B are said to be independent events, if the happening (or non-happening) of the other otherwise events are known as dependent events.

eg, If we draw a card from a pack of well shuffled cards and replace it before drawing the second card, the result of the second draw is independent of the first draw. But, however, if the first card drawn is not replaced the the second draw is denpendent on the first draw.

9. Complement of an Events

A sample space associated with a random experiment be S and event is E, then complement of an event E is denoted by E', Ec................. means event E does not occur.

eg, When an unbiased die is thrown, then the sample space

$$S = \{1,2,3,4,5,6\}$$

and $E = \{1,3,5\}$

then $E' = \{2,4,6\}$

✵ Probability

The probability of an event E to occur is the ratio of the number of cases in its favour to the total number of cases.

$$\therefore \qquad P(E) = \frac{\text{Number of cases favourable to event E}}{\text{Total number of cases}}$$

$$= \frac{n(E)}{n(S)}$$

Probability of non-occurrence of event E is

$$P(E) = 1 - P(E)$$

and $0 \le P(E) \le 1$. If $P(E) = 1$, then event E is known as certain event and if $P(E) = 0$, then E is known as impossible event.

Odds in favour and odds against an event If a is the number of cases favourale ot the event E, b is the number of cases favourable to the eventE. (ie, number of cases against to E) Then odds infavour of E are a:b and odds against of E are b:a. Then $P(E) = \frac{a}{a+b}$ and $P(E) = \frac{b}{a+b}$,

Thus, ods in favour of an event E = $\frac{a}{b} = \frac{a/(a+b)}{b/(a+b)} = \frac{P(E)}{P(E)}$

and odds against an event E = $\frac{b}{a} = \frac{P(E)}{P(E)}$.

Important Result Related to probability

(i) If E_1 and E_2 are events associated with a random experiment, then

$P(E_1 \cup E_2) = P(E_1) + P(E_2) - P(E_1 \cap E_2)$ (Addition theorem)

If E_1 and E_2 are mutually exclusive events, ie, $E_1 \cap E_2 = \phi$, then

$$P(E_1 \cup E_2) = P(E_1) + P(E_2) \qquad [\therefore P(\phi) = 0]$$

(ii) If E_1, E_2 and E_3 are events associated with a random experiment, then

$P(E_1 \cup E_2 \cup E_3) = P(E_1) + P(E_2) + P(E_3) - P(E_1 \cap E_2) - P(E_2 \cap E_3) - P(E_1 \cap E_3) + P(E_1 \cap E_2 \cap E_3)$

If E_1, E_2 and E_3 are mutually exclusive events, then

$P(E_1 \cup E_2 \cup E_3) = P(E_1) + P(E_2) + P(E_3)$

(iii) In general $P(E_1 \cup E_2 \cup \ldots\ldots \cup E_n)$

$$= \sum_{i=1}^{n} P(E_i) - \sum_{i<j} P(E_i \cap E_j) + \sum_{i<j<k} P(E_i \cap E_j \cap E_k) - \ldots\ldots + (-1)^{n-1} P(E_1 \cap E_2 \cap \ldots\ldots \cap E_n)$$

If $E_1, E_2, \ldots\ldots, E_n$ are mutually exclusive events, then

$$P(E_1 \cup E_2 \cup \ldots\ldots \cup E_n) = \sum_{i=1}^{n} P(E_i)$$

(iv) (a) $P(\overline{E}_1 \cap E_2) = P(E_2) - P(E_1 \cap E_2)$

(b) $P(E_1 \cap \overline{E}_2) = P(E_1) - P(E_1 \cap E_2)$

(v) If $E_2 \subset E_1$, then

(a) $P(E_1 \cap \overline{E}_2) = P(E_1) - P(E_2)$

(b) $P(E_2) \leq P(E_1)$

(vi) **Booley's inequality** If events $E_1, E_2, E_3, \ldots\ldots, E_n$ are associated with a random experiment, then

(a) $P\left(\bigcap_{i=1}^{n} E_i\right) \geq \sum_{i=1}^{n} P(E_i) - (n-1)$

(b) $P\left(\bigcap_{i=1}^{n} E_i\right) \leq \sum_{i=1}^{n} P(E_i)$

(vii) P(At least two events from E_1, E_2, E_3 occur)

$$= P(E_1 \cap E_2) + P(E_2 \cap E_3) + P(E_3 \cap E_1) - 2P(E_1 \cap E_2 \cap E_3)$$

(viii) P(Two events from E_1, E_2, E_3 occur)

$$= P(E_1 \cap E_2) + P(E_2 \cap E_3) + P(E_3 \cap E_1) - 3P(E_1 \cap E_2 \cap E_3)$$

(ix) P(Only one event occur from E_1, E_2, E_3)

$$= P(E_1) + P(E_2) + P(E_3) - 2P(E_1 \cap E_2) - 2P(E_2 \cap E_3) - 2P(E_3 \cap E_1) + 3P(E_1 \cap E_2 \cap E_3)$$

(x) If $E_1, E_2, E_3, \ldots\ldots\ldots\ldots, E_n$ are independent events, then

$$P(E_1 \cap E_2 \cap E_2 \cap \ldots\ldots \cap E_n) = P(E_1)\, P(E_2) \ldots\ldots\ldots\ldots\ldots\ldots P(E_n)$$

(xi) If $E_1, E_2, E_3, \ldots\ldots\ldots\ldots, E_n$ are exhaustive events, then

$$P(E_1 \cup E_2 \cup \ldots\ldots \cup E_n) = 1$$

(xii)If $E_1, E_2, E_3, \ldots\ldots\ldots\ldots, E_n$ are mutually exclusice and exhaustive events, then

$$P(E_1 \cup E_2 \cup \ldots\ldots \cup E_n) = P(E_1) + P(E_2) + \ldots\ldots\ldots\ldots\ldots + P(E_n) = 1$$

(xiii) If $E_1, E_2, E_3, \ldots\ldots\ldots\ldots, E_n$ are n events, then

(a) $P(E_1 \cup E_2 \cup E_3 \cup \ldots\ldots \cup E_n) \leq P(E_1) + P(E_2) + \ldots\ldots\ldots\ldots + P(E_n)$

(b) $P(E_1 \cup E_2 \cup \ldots\ldots \cup E_n) \geq 1 - P(\overline{E}_1) - P(\overline{E}_2) - \ldots\ldots\ldots\ldots - P(\overline{E}_n)$

❋ Conditional Probability

Events E_1 and E_2 are asscociated with a random experiment. Then, the probability of occurrence of event E_1 under the condition that E_2 has already occurred and $P(E_2) = 0$, is called the conditional probability and it is denoted by $P\left(\frac{E_1}{E_2}\right)$.

$$P\left(\frac{E_1}{E_2}\right) = \frac{P(E_1 \cap E_2)}{P(E_2)},\ E_2 \neq \phi$$

Important Result Related to Conditional Probability

(i) If E1 and E2 are independent events, then

$$P\left(\frac{E_1}{E_2}\right) = P(E_2)$$

(ii) If $E_1, E_2, E_3, \ldots\ldots, E_n$ are independendnt events, then

$$P(E_1 \cup E_2 \cup \ldots\ldots\ldots \cup E_n) = 1 - P(E'_1)\, P(E'_2) \ldots\ldots P(E'_n)$$

(iii) If E_1 and E_2 are events such that $E2 \neq 0$, then

$$P\left(\frac{E_1}{E_2}\right) + P\left(\frac{E'_1}{E_2}\right) = 1$$

(iv) If E_1 and E_2 are events such that $E_1 \neq \phi$, then

$$P(E_2) = P(E_1) \cdot P\left(\frac{E_2}{E_1}\right) + P(E_1) \cdot P\left(\frac{E_2}{E'_1}\right)$$

(v) If E_1, E_2 and E_3 are three events such that

$E_1 \neq \phi$, $E_1E_2 \neq \phi$, then

$$P(E_1 \cap E_2 \cap E_3) = P(E_1) \cdot P\left(\frac{E_2}{E_1}\right) \cdot P\left(\frac{E_3}{E_1E_2}\right)$$

In general, if $E_1, E_2, \ldots. E_n$ are n events such that $E_1 \neq \phi$, $E_1E_2 \neq \phi$, $E_1E_2E_3 \neq \phi, \ldots\ldots, E1E2\ldots\ldots En-1 \neq \phi$, then

$$P(E_1) \cdot P\left(\frac{E_2}{E_1}\right) \cdot P\left(\frac{E_3}{E_1E_2}\right) \ldots\ldots\ldots P\left(\frac{E_n}{E_1E_2\ldots\ldots E_{n-1}}\right)$$

(vi) If E_1 and E_2 are independent events, then

(a) $P(E_1 \cap E_2) = P(E_1) \cdot P\left(\frac{E_2}{E_1}\right)$, if $P(E_1) \neq 0$.

(b) $P(E_1 \cap E_2) = P(E_1) \cdot P\left(\frac{E_1}{E_2}\right)$, if $P(E_1) \neq 0$.

(vii) If E_1 and E_2 are independent events, then

$$P(E_1 \cap E_2) = P(E_1) \cdot P(E_2)$$

(viii) If E1, E2,......En are independent events, then

$$P(E_1 \cap E_2 \cap E_3 \cap E_4 \cap \ldots\ldots\ldots\ldots \cap E_n) = P(E_1)P(E_2) \ldots\ldots\ldots\ldots P(E_n)$$

✵ <u>Law of Total Probability</u>

Let in a random experiment S is a sample space and $E_1, E_2, \ldots\ldots, E_n$ are mutually exclusive and exhaustive events. If A is any event which occur with E_1 or E_2 or E_3 or..........or E_n, then

$$P(A) = P(E_1) \cdot P\left(\frac{A}{E_1}\right) + P(E_2) \cdot P\left(\frac{A}{E_2}\right) + ... + P(E_n) \cdot P\left(\frac{A}{E_n}\right) = \sum_{r=1}^{n} P(E_r) \cdot P\left(\frac{A}{E_r}\right)$$

Baye's Theorem

If $E_1, E_2, ..., E_n$ are mutually exclusive and exhaustive events with $P(E_i) = 0$, (i = 1, 2, ..., n), then for any event E which is a subset of E_i such that P(E) > 0, then

$$P\left(\frac{E_i}{E}\right) = \frac{P(E_i) \cdot P\left(\frac{E}{E_i}\right)}{\sum_{i=1}^{n} P(E_i) \cdot P\left(\frac{E}{E_i}\right)}$$

Random Variable and Its Distribution

Random Variable

Let S be the sample sapce associated with a given random experiment. Then, a real valued function X Which assigns to each event $w \in S$ to a uique real number X(w) is called a random variable.

A random variable is a function that associate a unique numerical value with every outcome of an experiment. The value of the random variable will vary from trial to trial as the experiment is repeated.

A random variable is usually denoted by the capital X, Y, Z,.....etc.

eg, A coin is tossed ten times. The random variable X is number of tails are noted. X can only take the values 0, 1, 2, ..., 10. So, X is a discrete random variable.

Probability Distribution

The probability distribution of a discrete of a discrete random variable X is a function which given the probability $P(x_i)$ that the random variable equals xi, for each value x_i.

$$P(x_i) = P(X = x_i)$$

It satisfies the following conditions.

(i) $0 \leq P(x_i) \leq 1$

(ii) $\Sigma P(x_i) = 1$

Mean and Variance of a Random Variable

If X is a discrete random variable which assumes values $x_1, x_2, x_3, ..., x_n$ with respective probabilites $p_1, p_2, p_3, ..., p_n$, then the mean $\overline{X}$ of X is defined as

$$\overline{X} = p_1 x_1 + p_2 x_2 + ... + p_n x_n$$

or $$\overline{X} = \sum_{i=1}^{n} p_i x_i$$

and variance of X is defined as

$$\text{ver}(X) = p_1(x_1 - \overline{X})^2 + p_2(x_2 - \overline{X})^2 + \dots\dots\dots + p_n(x_n - \overline{X})^2 = \sum_{i=1}^{n} p_i(x_i - \overline{X})^2$$

where, $\overline{X} = \sum_{i=1}^{n} p_i x_i$ is the mean of X.

or $\quad \text{ver}(X) = \sum_{i=1}^{n} p_i x_i^2 - \left(\sum_{i=1}^{n} p_i x_i\right)^2$

The square root of the variance gives the standard deviation

ie,

$$\sqrt{\text{var}(X)} = \sqrt{\sigma^2} = \sigma$$

✵ Random Variable and Its Distribution

Let a binomial experiment has probability of success p and that of failure q (ie, p + q = 1). If E be an event and let X = number of sucess ie, number of times event E occurs in n trials.

Then probability distribution of binomial distution with parameters n and p is given by

P(X = r) = probability of r success in n trials

$= {}^nC_r\, p^r q^{n-r}$ (p + q = 1)

$=$ (r + 1)th term in the expansion of $(q + n)^n$.

It is written as X ~ B(n, p) of X ~ B*i*(n, p).

Mean and Variance of Binomial Distribution

Let X ~ B (n, p), then

$$P(X = r) = {}^nC_r\, p^r q^{n-r}$$

where, r = 0, 1, 2,......., n and p + q = 1

$\therefore$ Mean, $\overline{X} = E(X) = np$

and variance, var(X) = npq

Standard deviation = $\sqrt{npq}$

Relation between Mean and Variance

Mean – Variance = np – npq = np(1 – q) = $np^2 > 0$

$\Rightarrow$ Mean > Variance

ie, For binomial variable X, value of mean is always greater then its variance.

Mode of Binomial Distribution

In binomial distribution, the value of r for which P(X = r) is maximum, is known as mode of binomial distribution.

$\therefore \quad (n + 1)p - 1 \le r \le (n + 1)P$

✵ Poisson Distribution

It is limiting case of binomial distribution. Let the number of events n is large ($n \to \infty$) and probability of success in each experiment is 0 and np = λ (say) is finite, then

$$P(X = r) \quad \text{or} \quad P(r) = \frac{e^{-\lambda}\lambda^r}{r!}, \qquad \text{where } r = 0, 1, 2, ...$$

and l = np. Here, l is known as parameter of poisson distribution.

$P(r + 1) = \dfrac{\lambda}{r+1} P(r)$ is known as recurrence formula.

Note :

- The result of a random expriment is called an outcome.
- If $E_1, E_2, ..., E_n$ are mutually exclusive events, then $E_1 \cap E_2 \cap .... \cap E_n = \phi$
- In throwing of n dice, the exhaustive number of case is 6^n.
- If E_1 and E_2 are independent events, then

  E_1 and $\overline{E}_2$ are independent events

  $\overline{E}_1$ and E_2 are independent events

  $\overline{E}_1$ and $\overline{E}_2$ are independent events

  Let the sample space associated to a random experiment is S, then ϕ and S are subset of S. the event ϕ is known as impossible event and event S is known as certain event.
- The probabilites $P(E_1), P(E_2), ...P(E_i)$ are called 'prior probabilites' because they exist before we gain any information from the experiment itself.
- The probabilites $P\left(\frac{E_i}{E}\right)$, i = 1, 2,......,n are called 'posterior probabilities' because they are determined after the result of the experiment are known.
- A random variable which can take only finite or countably infinite number of vales is called a discrete random variable.
- The mean of a random variable X is also known as its mathematical expectation or expected value and is denoted by E(X).
- The variance and standard deviation of a random variable are always non-negative.
- The trials mush meet the following requirements

  the total number of trials is fixed in advance.

  there are just two outcomes of each trials, sucess and failure

  all the trials have the same probability of success.

  the outcomes of all the trial are statistically independent.
- $\sum_{r=0}^{n} P(r) = 1$
- If λ_1 and λ_2 are parameter of variables x and y, then parameter of (x + y) will be $(\lambda_1 + \lambda_2)$.
- In poisson distribution, mean = variance = λ = np.