MATHEMATICS

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AIM POINT
MATHEMATICS
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PROPERTIES OF TRIANGLES

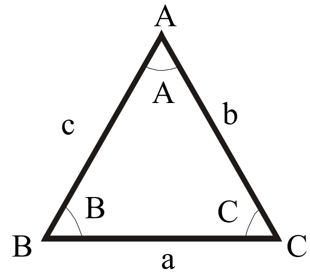
& Their Properties

CONTENTS

THINGS TO REMEMBER

✷ **Relation between the Sides and Angles of Triangle :**

In a ΔABC, angles are denoted by A, B and C the lengths of corresponding sides opposite to these angles are denoted by a, b and c respectively. Area and perimeter of a tringle are denoted by and 2s respectively.



Also,

Semi-perimeter of the triangle is

$$s = \frac{a+b+c}{2}$$

Sine Rule

In any ΔABC, the sines of the angles are proportional to the lengths of the opposite side,

ie,

$$\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}$$

It can also be written as

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} = k \qquad \text{(say)}$$

then a = k sin A, b = k sin B, c = k sin C

Cosine Rule

In any ΔABC cosine of an angle can express in terms of sides.

$$\cos A = \frac{b^2 + c^2 - a^2}{2bc}$$

$$\cos B = \frac{c^2 + a^2 - b^2}{2bc}$$

and

$$\cos C = \frac{a^2 + b^2 - a^2}{2bc}$$

Projection Rule

In any ΔABC

a = b cos C + c cos B

b = a cos C + c cos A

c = a cos B + b cos A

Napier's Rule

In any ΔABC

$$\tan \frac{C-A}{2} = \frac{c-a}{c+a} \cot \frac{B}{2}$$

$$\tan \frac{A-B}{2} = \frac{a-b}{a+b} \cot \frac{C}{2}$$

$$\tan\frac{B-C}{2}=\frac{b-c}{b+c}\cot\frac{A}{2}$$

✷ Trigonometric Ratios of Half Angles of Any Triangle :

In a ΔABC, if the sides of the triangle are a,b,c and corresponding angles are A,B,C respectively and s is the semi-perimeter, then

1. $\sin\frac{A}{2}=\sqrt{\frac{(s-b)(s-c)}{bc}}$

2. $\sin\frac{B}{2}=\sqrt{\frac{(s-c)(s-a)}{ca}}$

3. $\sin\frac{C}{2}=\sqrt{\frac{(s-a)(s-b)}{ab}}$

4. $\cos\frac{A}{2}=\sqrt{\frac{s(s-a)}{bc}}$

5. $\cos\frac{B}{2}=\sqrt{\frac{s(s-b)}{ca}}$

6. $\cos\frac{C}{2}=\sqrt{\frac{s(s-c)}{ab}}$

7. $\tan\frac{A}{2}=\sqrt{\frac{(s-b)(s-c)}{s(s-a)}}$

8. $\tan\frac{B}{2}=\sqrt{\frac{(s-a)(s-c)}{s(s-b)}}$

9. $\tan\frac{C}{2}=\sqrt{\frac{(s-a)(s-b)}{s(s-c)}}$

✷ Area of Tringles :

In a ΔABC, if the sides of the tringle are a, b, c and corresponding angles are A, B, C respectively, then area of tringle.

When two sides and Angles between Them is Known.

$$\Delta=\frac{1}{2}ab\sin C$$

$$\Delta=\frac{1}{2}bc\sin A$$

$$\Delta=\frac{1}{2}ca\sin B$$

When One Side and Corresponding Angles are Known

$$\Delta = \frac{c^2 \sin A \sin B}{2\sin C}$$

$$\Delta = \frac{a^2 \sin B \sin C}{2\sin A}$$

$$\Delta = \frac{b^2 \sin C \sin A}{2\sin B}$$

When all the Three Sides are Known

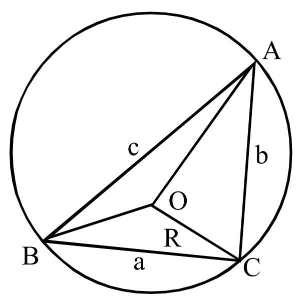
$$\Delta = \sqrt{s(s-a)(s-b)(s-c)}$$

It is Known as Hero's formula.

✱ **<u>Different Types of Circle Connected with Tringles :</u>**

1. **Circumcircle of a Triangle**

The circle passing through the vertices of a ΔABC is called circumcircle. Its radius R is called the circumradius and its centre is known as circumcentre. Circumcentre is the point of intersection of perpendicular bisectors of the sides.

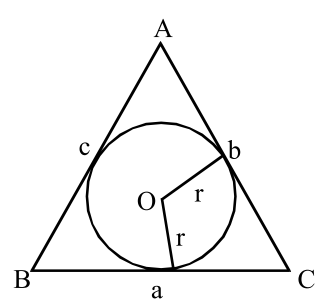


1. $R = \frac{a}{2\sin A} = \frac{c}{2\sin C}$

2. $R = \frac{abc}{4\Delta}$

2. **Incircle of a Triangle**

The circle touching all the sides of a trinangle internally is known as an incircle of the triangle. Its centre is called incentre and its radius r is called the inradius of the circle. Incenter is the point of intersection of besectors of the angles of the triangle.



1. $r = \frac{\Delta}{s}$

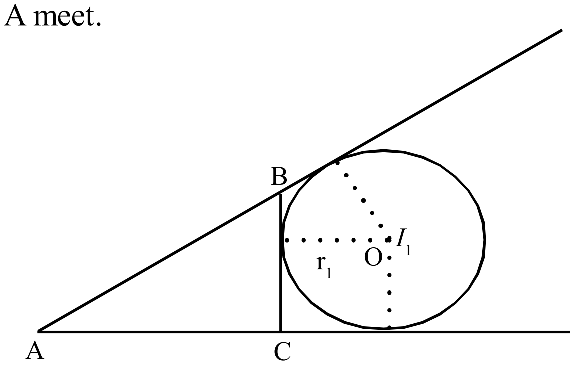
2. $r = (s-a)\tan\frac{A}{2} = (s-b)\tan\frac{B}{2} = (s-c)\tan\frac{C}{2}$

3. $r = \frac{a\sin\frac{B}{2}\sin\frac{C}{2}}{\cos\frac{A}{2}} = \frac{b\sin\frac{A}{2}\sin\frac{C}{2}}{\cos\frac{B}{2}} = \frac{c\sin\frac{B}{2}\sin\frac{A}{2}}{\cos\frac{C}{2}}$

4. $r = 4R\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}$

3. Escribed Circles of a Triangle

The circle touching BC and the two sides AB and AC produced of ΔABC externally is called the escribed circle opposite to A. Its radius is denoted by r_1. The cente of this circle is known as excenter and denoted by *I*. It is the point where the external bisectors of the angle B and C and the interval bisector of the angle A meet.



Similarly, r_2 and r_3 denote the radii of the escribed circles opposite to the angles B and C respectively and excentres are denoted by I_2 and I_3. r_1, r_2, r_3 are called the exradiof ΔABC. Here,

1. $r_1 = \dfrac{\Delta}{s-a} = s\tan\dfrac{A}{2} = 4R\sin\dfrac{A}{2}\cos\dfrac{B}{2}\cos\dfrac{C}{2} = \dfrac{a\cos\dfrac{B}{2}\cos\dfrac{C}{2}}{\cos\dfrac{A}{2}}$

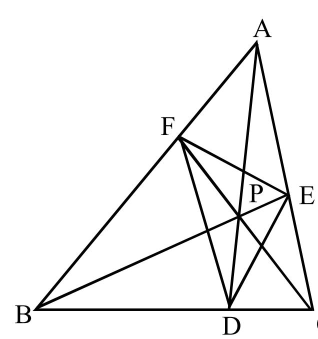
2. $r_2 = \dfrac{\Delta}{s-b} = s\tan\dfrac{B}{2} = 4R\sin\dfrac{B}{2}\cos\dfrac{C}{2}\cos\dfrac{A}{2} = \dfrac{b\cos\dfrac{A}{2}\cos\dfrac{C}{2}}{\cos\dfrac{B}{2}}$

3. $r_3 = \dfrac{\Delta}{s-c} = s\tan\dfrac{C}{2} = 4R\sin\dfrac{C}{2}\cos\dfrac{A}{2}\cos\dfrac{B}{2} = \dfrac{c\cos\dfrac{A}{2}\cos\dfrac{B}{2}}{\cos\dfrac{C}{2}}$

✱ **<u>Orthocentre of a Tringle. :</u>**

In a ΔABC, AD, BE and CF are perpendiculars from the vertices A, B and C respectively to the opposite sides. These three perpendiculars are concurrent at the point P which is called the orthocentre of the ΔABC. The ΔDEF is called the pedal triangle of ΔABC

1. PA = 2R cos A, PB = 2R cos B, PC = 2R cos C
2. PD = 2R cos B cos C, PE = 2R cos C cos A, PF = 2R cos A cos B

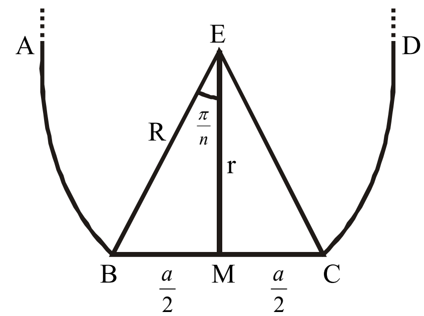


✷ Regular Polygon :

A regular polygon is a polygon which has all its sides as well as all its angles are equal. The circle passing through all the vertices of a regular polygon is called its circumscribed circle and the circle which touches all the sides of a regular polygon is called its inscribed circle.

Radius of circumcircle, $R = \frac{a}{2}\operatorname{cosec}\frac{\pi}{n}$

and radius of incircle, $r = \frac{a}{2}\cot\frac{\pi}{n}$



Where, a is the length of the side of a regular polygon.

$$\text{Area of a regular polygon} = \frac{1}{4}na^2\cot\frac{\pi}{n}$$

$$= nr^2\tan\frac{\pi}{n}$$

$$= \frac{n}{2}R^2\sin\frac{2\pi}{n}$$

✷ Solutions of Triangles :

The three sides a, b, c and the three angles A, B, C are called the elements of the ABC. When any three of these six elements (except all the three angles) of a triangle are given, the triangle is known as completely. That is the other three elements can be expressed in terms of the given elements and can be evaluated. this process is called the solution of triangles.

Solution of right angled Triangle

Let the triangle be right angled at C, then

(i) When two sides are given if a, b are given, then $\tan A = \frac{a}{b}, B = 90^o - A, C = \frac{a}{\sin A}$

if a, c are given then $\sin A = \frac{a}{c}, b = c\cos A, B = 90^o - A$

(ii) When one side and an acute angle are given If a, A are given, then $B = 90^o - A, b = a\cot A, c = \frac{a}{\sin A}$

if c, A are given, then $B = 90^o - A$, $a = c\sin A$, $b = c\cos A$

Solution of a Triangle in General

(i) If the three sides a, b, c, are given, then Angle A is obtained from

$$\tan\frac{A}{2} = \sqrt{\frac{(s-b)(s-c)}{s(s-a)}}$$

or

$$\cos A = \frac{b^2+c^2-a^2}{2bc}$$

(ii) If two sides b and c and the included angle A are given, then

$$\tan\frac{B-C}{2}=\frac{b-c}{b+c}\cot\frac{A}{2} \text{ gives } \frac{B-C}{2}$$

Also, $\frac{B+C}{2}=90^o-\frac{A}{2}$, so that B and C can be evaluated. The third side is obtained either by sine rule of cosine rule.

(iii) When one side 'a' and two angles (∠A and ∠B) are given, then $\angle C = 180^o - (\angle A + \angle B)$ and the other sides can be determined by sine rule.

(iv) When two sides and angle opposite to one of them be given In this case, the triangle is not always uniquely determined. It is quite possible to have no triangle, one triangle and two triangles with this type of data. So, it is called an ambiguous case.

By Cosine Law

$$b^2 = c^2 + a^2 - 2ac \cos B$$

$\Rightarrow a^2 - 2ac \cos B = b^2 - c^2$

$\Rightarrow a^2 - 2ac \cos B = c^2 \cos^2 B = b^2 - c^2 + c^2 \cos^2 B$

$\Rightarrow (a - c \cos B)^2 = b^2 - c^2 (1 - \cos^2 B) = b^2 - c^2 \sin^2 B$

$\Rightarrow (a - c \cos B) = \pm\sqrt{b^2-c^2\sin^2 B}$

$\Rightarrow a = c \cos B \pm\sqrt{b^2-c^2\sin^2 B}$

This equation helps us to determine a, when b, c and B are being given. Now tha following cases arise.

Case I. When $b < c \sin B$, then a is imaginary and so there is no solution.

Case II.

When $b = c \sin B$

then $a = c \cos B$,

so there is unique solution

Also, $b = c \sin B$

$\Rightarrow k\sin C=\frac{k\sin B}{\sin B}$

$\Rightarrow \sin C = 1$

$\Rightarrow \angle C = 90^o$

The triangle is right angled in this case.

Case III.

When $b > c \sin B$,

then $b^2 - c^2 \sin^2 B > 0$,

then there are two solutions given by

$$a=c\cos B+\sqrt{b^2-c^2\sin^2 B}$$

and

$$a=c\cos B-\sqrt{b^2-c^2\sin^2 B}$$

Now, if B is an acute angle, then there are two triangles proided that $c > b > c \sin B$ and if B is an abtuse angle, then there is only one triangle provided that $b > c$.

Note :

- $r_1 + r_2 + r_3 = 4R + r$
- $r_1r_2 + r_2r_3 + r_3r_1 = s^2 = \dfrac{r_1r_2r_3}{r}$
- $\dfrac{1}{r_1}+\dfrac{1}{r_2}+\dfrac{1}{r_3}=\dfrac{s}{\Delta}=\dfrac{1}{r}$ and $r_1r_2r_3 = r^2\left(\cot\dfrac{A}{2}\cot\dfrac{B}{2}\cot\dfrac{C}{2}\right)^2$
- If p1, p2, p3 are perpendicular's drawn from the vertices to opposite side of the triangle, then

  (a) $\dfrac{1}{p_1}+\dfrac{1}{p_2}+\dfrac{1}{p_3}=\dfrac{1}{r}$

  (b) $p_1p_2p_3 = \dfrac{(abc)^2}{8R^2}$

  (c) $\dfrac{\cos A}{p_1}+\dfrac{\cos B}{p_2}+\dfrac{\cos C}{p_3}=\dfrac{1}{R}$
- If the polygon has n sides, then sum of the internal angles is $(n - 2)\pi$ and each angle is $\dfrac{(n-2)\pi}{n}$.
- If B is acute and c < b, then there is only one triangle exist.
- If B is obtuse and b < c, no such triangle exists.
- B is an acute angle if cos B is positive and if cos B is negative, then B is an obtuse angle.