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VECTORS

IMPORTANT FORMULAE

- The sum of two vectors $\overrightarrow{AB}$ and $\overrightarrow{BC}$ is $\overrightarrow{AB} + \overrightarrow{BC} = \overrightarrow{AC}$.
- $\overrightarrow{AB}$ = Position vector of $\overrightarrow{B}$-position vector of $\overrightarrow{A}$
- If the position vectors of the points A and B are $\vec{a}$ and $\vec{b}$ and a point P divides the line AB in the ratio of $m : n$ then position vector of P = $\dfrac{n\vec{a} + m\vec{b}}{m+n}$.
- Position vector of mid-point of AB = $\dfrac{\vec{a} + \vec{b}}{2}$
- Position vector of point P (a, b), $\overrightarrow{OP} = a\hat{i} + b\hat{j}$ and modulus of vector $\overrightarrow{OP}$;

$$OP = \sqrt{a^2 + b^2}$$

- Unit vector = $\dfrac{\text{Vector}}{\text{Modulus of vector}}$
- If the co-ordinates of a point P are (x, y, z), then position vector of point P,

$$\overrightarrow{OP} = x\hat{i} + y\hat{j} + z\hat{k}$$

and modulus of vector $\overrightarrow{OP} = \sqrt{x^2 + y^2 + z^2}$

- Scalar product of unit vectors $\hat{i}, \hat{j}, \hat{k}$ are given by $\hat{i}.\hat{i} = 1$, $\hat{j}.\hat{j} = 1$, $\hat{k}.\hat{k} = 1$, $\hat{i}.\hat{j} = 0$, $\hat{j}.\hat{k} = 0$, $\hat{k}.\hat{i} = 0$.
- If θ is the angle between two vectors $\vec{a}$ and $\vec{b}$, then

$$\cos\theta = \frac{\vec{a}.\vec{b}}{ab} = \frac{\vec{a}.\vec{b}}{|\vec{a}|.|\vec{b}|}$$

- Two vectors $\vec{a}$ and $\vec{b}$ are perpendicular if $\vec{a}.\vec{b} = 0$.
- Work done by any force
  = Force vector × Distance vector
- Vector product of two vector quantities $\vec{a}$ and $\vec{b}$ defined as

$$\vec{a} \times \vec{b} = (|\vec{a}|\ |\vec{b}| \sin\theta)\ \hat{n} = (ab\sin\theta)\hat{n},$$

where $\hat{n}$ is a unit vector perpendicular to each of $\vec{a}$ and $\vec{b}$.

- If $\vec{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$ and $\vec{b} = b_1\hat{i} + b_2\hat{j} + b_3\hat{k}$, then

$$\vec{a} \times \vec{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$$

Multiple Choice Questions

1. If $\overrightarrow{AO} + \overrightarrow{OB} = \overrightarrow{BO} + \overrightarrow{OC}$, then A, B, C are (*BSEB*, 2013)

(a) coplanar (b) collinear
(c) non-collinear (d) non-coplanar

2. The projection of the vector $\hat{i} - 2\hat{j} + \hat{k}$ on the vector $4\hat{i} - 4\hat{j} + 7\hat{k}$ is (*BSEB*, 2013)

(a) $\frac{19}{8}$ (b) $\frac{19}{9}$ (c) $\frac{19}{11}$ (d) $\frac{19}{7}$

3. The modulus of the vector $2\hat{i} - 7\hat{j} - 3\hat{k}$ is (*BSEB*, 2013)

(a) $\sqrt{61}$ (b) $\sqrt{62}$ (c) $\sqrt{64}$ (d) $\sqrt{32}$

4. The projection of the vector $2\hat{i} - \hat{j} - 1\hat{k}$ on the vector $\hat{i} - 2\hat{j} + 1\hat{k}$ is (*BSEB*, 2013)

(a) $\frac{4}{\sqrt{6}}$ (b) $\frac{5}{\sqrt{6}}$ (c) $\frac{4}{\sqrt{3}}$ (d) $\frac{7}{\sqrt{6}}$

5. If $\vec{a} = \hat{i} + \hat{j} + 2\hat{k}$ and $\vec{b} = 3\hat{i} + 2\hat{j} - \hat{k}$ then the value of $(\vec{a} + 3\vec{b}).(2\vec{a} - \vec{b})$ = (*BSEB*, 2013)

(a) 15 (b) – 15 (c) 18 (d) – 18

6. If $|\vec{a}| = \sqrt{26}$, $|\vec{b}| = 7$ and $|\vec{a} \times \vec{b}| = 35$, then $\vec{a}.\vec{b}$ = (*BSEB*, 2013)

(a) 8 (b) 7 (c) 9 (d) 12

7. If $\vec{a} = 2\hat{i} - 3\hat{j} + 4\hat{k}$ and $\vec{b} = \hat{i} + 2\hat{j} + \hat{k}$ then $\vec{a} + \vec{b}$ = (*BSEB*, 2012)

(a) $\hat{i} + \hat{j} + 3\hat{k}$ (b) $3\hat{i} - \hat{j} + 5\hat{k}$
(c) $\hat{i} - \hat{j} - 3\hat{k}$ (d) $2\hat{i} + \hat{j} + \hat{k}$

8. If $\vec{a} = \hat{i} + 2\hat{j} + 3\hat{k}$ and $\vec{b} = 3\hat{i} + 2\hat{j} + \hat{k}$ then $\cos\theta$ = (*BSEB*, 2012)

(a) $\frac{6}{7}$ (b) $\frac{5}{7}$ (c) $\frac{4}{7}$ (d) $\frac{1}{2}$

9. If $|\vec{a}+\vec{b}| = |\vec{a}-\vec{b}|$ then (BSEB, 2012)

(a) $\vec{a} \,||\, \vec{b}$ (b) $\vec{a} \perp \vec{b}$

(c) $|\vec{a}| = |\vec{b}|$ (d) none of these

10. The angle between the vectors $2\hat{i}-3\hat{j}+2\hat{k}$ and $\hat{i}+4\hat{j}+5\hat{k}$ is given by (BSEB, 2011)

(a) 30° (b) 90° (c) 45° (d) 60°

11. The position vector of the point (1, 0, 2) is (BSEB, 2015)

(a) $\vec{i}+\vec{j}+2\vec{k}$ (b) $\vec{i}+2\vec{j}$

(c) $\vec{i}+3\vec{k}$ (d) $\vec{i}+2\vec{k}$

12. The modulus of $7\vec{i}-2\vec{j}+\vec{k}$ is (BSEB, 2015)

(a) $\sqrt{10}$ (b) $\sqrt{55}$ (c) $3\sqrt{6}$ (d) 6

13. If O be the origin and $\vec{OP} = 2\vec{i}+3\vec{j}-4\vec{k}$ and $\vec{OQ} = 5\vec{i}+4\vec{j}-3\vec{k}$, then $\vec{PQ}$ is equal to (BSEB, 2015)

(a) $7\vec{i}+\vec{j}-7\vec{k}$ (b) $-3\vec{i}-\vec{j}-\vec{k}$

(c) $-7\vec{i}-7\vec{j}+7\vec{k}$ (d) $3\vec{i}+\vec{j}+\vec{k}$

14. The scalar product of $5\hat{i}+\hat{j}-3\hat{k}$ and $3\hat{i}-4\hat{j}+7\hat{k}$ is (BSEB, 2015)

(a) 10 (b) – 10 (c) 15 (d) – 15

15. If $\vec{a}.\vec{b} = 0$, then (BSEB, 2015)

(a) $\vec{a} \perp \vec{b}$ (b) $\vec{a} \,||\, \vec{b}$

(c) $\vec{a}+\vec{b}=0$ (d) $\vec{a}-\vec{b}=0$

16. $\vec{i}.\vec{j} = 0$ (BSEB, 2015)

(a) 0 (b) 1 (c) $\vec{k}$ (d) $-\vec{k}$

17. $\vec{k}\times\vec{j} = 0$ (BSEB, 2015)

(a) 0 (b) 1 (c) $\vec{i}$ (d) $-\vec{i}$

18. $\vec{a}.\vec{a} =$ (BSEB, 2015)

(a) 0 (b) 1 (c) $|\vec{a}|^2$ (d) $|\vec{a}|$

Ans. 1. (b), 2. (b), 3. (b), 4. (b), 5. (b), 6. (b), 7. (b), 8. (b), 9. (b), 10. (b), 11. (d), 12. (c), 13. (d), 14. (b), 15. (a), 16. (a), 17. (d), 18. (b).

Very Short Answer Type Questions

Q. 1. Find $[\vec{i}\ \vec{j}\ \vec{k}]$ where $\vec{i}, \vec{j}$ and $\vec{k}$ are mutually perpendicular unit vectors. (BSEB, 2014)

Solution :

$$[\vec{i}\ \vec{j}\ \vec{k}] = \begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix} = 1$$

Q. 2. Prove that one vectors $\vec{i}-2\vec{j}+5\vec{k}$ and $-2\vec{i}+4\vec{j}+2\vec{k}$ are mutually perpendicular. (BSEB, 2010)

Solution :

$$(\vec{i}-2\vec{j}+5\vec{k}).(-2\vec{i}+4\vec{j}+2\vec{k}) = (1)(-2)+(-2)(4)+(5)(2) = -2-8+10 = 0$$

$$\therefore (\vec{i}-2\vec{j}+5\vec{k}) \perp (-2\vec{i}+4\vec{j}+2\vec{k})$$

Q. 3. Find the projection of vector $\vec{a} = 2\vec{i}+3\vec{j}+2\vec{k}$ on a vector $\vec{a} = \hat{i}+2\hat{j}+\vec{k}$. (BSEB, 2013)

Solution :

$$\vec{a}.\vec{b} = (2\vec{i}+3\vec{j}+2\vec{k}).(\vec{i}+2\vec{j}+\vec{k}) = (2)(1)+(3)(2)+(2)(1) = 10$$

$$|\vec{b}| = \sqrt{(1)^2+(2)^2+(1)^2} = \sqrt{6}$$

$\therefore$ Required projection

$$= \frac{\vec{a}.\vec{b}}{|\vec{b}|} = \frac{10}{\sqrt{6}}$$

Q. 4. For the vector $\vec{a}$ and $\vec{b}$ prove that $|\vec{a}\times\vec{b}|^2 = |\vec{a}|^2|\vec{b}|^2 - 1\,|\vec{a}.\vec{b}|^2$ (BSER, 2013)

Solution :

Let $|\vec{a}| = a$, $|\vec{b}| = b$.

Let the angle between $\vec{a}$ and $\vec{b}$ be θ, then

$$\text{LHS} = |\vec{a}\times\vec{b}|^2 = |ab\sin\theta|^2 = a^2b^2\sin^2\theta$$

$$\text{RHS} = |\vec{a}|^2|\vec{b}|^2 - |\vec{a}.\vec{b}|^2 = a^2b^2-(ab\cos\theta)^2 = a^2b^2(1-\cos^2\theta) = a^2b^2\sin^2\theta \quad ...(2)$$

From (1) and (2),

LHS = RHS

Q. 5. Find unit vector of a given vector $\vec{a} = 2\hat{i}-3\hat{j}+4\hat{k}$. (*BSER*, 2014)

Solution :

$$|\vec{a}| = \sqrt{(2)^2+(-3)^2+(4)^2} = \sqrt{4+9+16} = \sqrt{29}$$

∴ Required unit vector

$$= \frac{\vec{a}}{|\vec{a}|} = \frac{2\hat{i}-3\hat{j}+4\hat{k}}{\sqrt{29}} = \frac{2}{\sqrt{29}}\hat{i}-\frac{3}{\sqrt{29}}\hat{j}+\frac{4}{\sqrt{29}}\hat{k}$$

Q. 6. Prove that the vectors $\vec{i}-2\vec{j}+5\vec{k}$ and $-2\vec{i}+4\vec{j}+2\vec{k}$ are mutually perpendicular. (*BSER*, 2013)

Solution

$$(\vec{i}-2\vec{j}+5\vec{k})\cdot(-2\vec{i}+4\vec{j}+2\vec{k}) = (1)(-2)+(-2)(4)+(5)(2) = -2-8+10 = 0$$

$$\therefore\ (\vec{i}-2\vec{j}+5\vec{k}) \perp (-2\vec{i}+4\vec{j}+2\vec{k})$$

Q. 7. If $\vec{a} = 2\vec{i}-3\vec{k}-5\vec{k}$ and $\vec{b} = -7\vec{i}+6\vec{j}+8\vec{k}$, find $\vec{a}\times\vec{b}$ (*JAC.*, 2013)

Solution :

$$\vec{a}\times\vec{b} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 2 & -3 & -5 \\ -7 & 6 & 8 \end{vmatrix}$$

$$= (-24+30)\vec{i}+(35-16)\vec{j}+(12-21)\vec{k}$$

$$= 6\vec{i}+19\vec{j}-9\vec{k}$$

Q. 8. If $\vec{a} = 2\vec{i}-\vec{j}$, then find the value of $\vec{a}.\vec{i}-\vec{a}.\vec{j}$. (*JAC.*, 2014)

Solution :

$$\vec{a}.\hat{i}-\vec{a}.\hat{j} = (2\hat{i}-\hat{j}).\hat{i}-(2\hat{i}-\hat{j}).\hat{j} = 2-(-1) = 3$$

Q. 9. Find a vector perpendicular to both $\vec{a} = 3\hat{i}-2\hat{j}+\hat{k}$ and $\vec{b} = \hat{i}+4\hat{j}-\hat{k}$. (*JAC.*, 2014)

Solution :

Required vector

$$= \vec{a}\times\vec{b} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 3 & -2 & 1 \\ 1 & 4 & -1 \end{vmatrix}$$

$$= (2-4)\vec{i}+(1+3)\vec{j}+(12+2)\vec{k}$$

$$= -2\hat{i}+4\hat{j}+14\hat{k}$$

Q. 10. Two vectors are $\vec{a} = 2\hat{i}+\hat{j}$ and $\hat{b} = 2\hat{i}+\hat{j}$, is $|\vec{a}|=|\vec{b}|$? Are the vectors $\vec{a}$ and $\vec{b}$ are equal. (*USEB*, 2013)

Solution :

$$|\vec{a}| = \sqrt{(1)^2+(2)^2} = \sqrt{5}$$

$$|\vec{b}| = \sqrt{(2)^2+(1)^2} = \sqrt{5}$$

$$\therefore\ |\vec{a}| = |\vec{b}|$$

But $\vec{a} \neq \vec{b}$ as the cofficients of $\hat{i}$ and $\hat{j}$ are unequal.

Q. 11. Find the angle between two vectors $\vec{a}$ and $\vec{b}$ with magnitudes 1 and 2 respectively and $\vec{a}.\vec{b}=1$. (*USEB*, 2013)

Solution :

$$\vec{a}.\vec{b} = ab\cos\theta$$

$$\Rightarrow\ 1 = (1)(2)\cos\theta$$

$$\Rightarrow\ 4\cos\theta = \frac{1}{2} = \cos 60^\circ$$

$$\Rightarrow\ \theta = 60^\circ$$

Q. 12. Find a vector in the direction of vector $\vec{a} = 3\vec{i}-4\vec{j}$ that has magnitude 5 units. (*USEB*, 2013)

Solution :

Required vector

$$= 5\times\frac{3\hat{i}-4\hat{j}}{\sqrt{(3)^2+(4)^2}} = 3\hat{i}-4\hat{j}$$

Q. 13. Find the value of $\hat{i}\ (\hat{j}\times\hat{k})+\hat{j}\ (\hat{i}\times\hat{k})+\hat{k}\ (\hat{i}\times\hat{j})$. (*USEB*, 2014)

Solution :

$$\hat{i}\cdot(\hat{j}\times\hat{k})+\hat{j}\,(\hat{i}\times\hat{k})+\hat{k}\,(\hat{i}\times\hat{j})$$
$$= \hat{i}.\hat{i}+\hat{j}\cdot(-\hat{j})+\hat{k}.\hat{k}$$
$$= 1-1+1$$
$$= 1$$

Q. 14. If $\vec{a} = x\hat{i}+2\hat{j}-3\hat{k}$ and $\vec{b} = 3\hat{i}-y\hat{j}+\hat{k}$ are two equal vectors, then find the one value of $x+y+z$. (*CBSE*, 2013)

Solution :

$$\vec{a} = \vec{b}$$
$\Rightarrow$ $x\hat{i}+2\hat{j}-z\hat{k} = 3\hat{i}-y\hat{j}+\hat{k}$

$\Rightarrow$ $x = 3$ (Equating the coefficients of $\hat{i}$, $\hat{j}$

$-y = 2$ and $\hat{k}$ on both sides)

$-z = 1$

$\Rightarrow$ $x = 3$

$y = -2$

and $z = -1$

$\therefore x+y+z = 0$

Q. 15. Write a unit vector in the direction of the sum of the vectors :

$\vec{a} = 2\hat{i}-\hat{j}+2\hat{k}$ and $\vec{b} = -\hat{i}+\hat{j}+3\hat{k}$. (*CBSE*, 2013)

Solution :

$$\vec{a}+\vec{b} = (2\hat{i}-\hat{j}+2\hat{k})+(-\hat{i}+\hat{j}+3\hat{k})$$
$\Rightarrow$ $\vec{a}+\vec{b} = \hat{i}+5\hat{k}$

$$|\vec{a}+\vec{b}| = \sqrt{(1)^2+(5)^2} = \sqrt{26}$$

$\therefore$ Required unit vector

$$= \frac{\vec{a}+\vec{b}}{|\vec{a}+\vec{b}|}$$
$$= \frac{\hat{i}+5\hat{k}}{\sqrt{26}}$$
$$= \frac{1}{\sqrt{26}}\hat{i}+\frac{5}{\sqrt{26}}\hat{k}$$

Q. 16. Find $|\vec{x}|$, if for a unit vactor $\vec{a}$, $(\vec{x}-\vec{a}).(\vec{x}\times\vec{a}) = 15$. (*AI CBSE*, 2013)

Solution :

$$(\vec{x}-\vec{a}).(\vec{x}+\vec{a}) = 15$$
$\Rightarrow$ $\vec{x}^2-\vec{a}^2 = 15$

$\Rightarrow$ $|\vec{x}|^2-|\vec{a}|^2 = 15$

$\Rightarrow$ $|\vec{x}|^2-1^2 = 15$

$\Rightarrow$ $|\vec{x}|^2 = 16$

$\Rightarrow$ $|\vec{x}| = 4$

Q. 17. Find the value of p for which one vectors $3\hat{i}+2\hat{j}+a\hat{k}$ and $\hat{i}-2p\hat{j}+3\hat{k}$ one parallel. (*AI CBSE*, 2014)

Solution :

For given vectors to be parallel

$$\frac{3}{1} = \frac{2}{-2p} = \frac{9}{3}$$
$\Rightarrow$ $3 = -\frac{1}{p} = 3$

$\Rightarrow$ $p = -\frac{1}{3}$

Q. 18. If vectors a and b are such that $|\vec{a}| = 3$, $|\vec{b}| = \frac{2}{3}$ and $\vec{a}\times\vec{b}$ is a unit vector, then write the angle between $\vec{a}$ and $\vec{b}$. (*CBSE*, 2014)

Solution :

$$|\vec{a}\times\vec{b}| = |\vec{a}||\vec{b}|\sin\theta$$
$\Rightarrow$ $1 = 3.\frac{2}{3}\sin\theta$

$\Rightarrow$ $\sin\theta = \frac{1}{2} = \sin 30°$

$\Rightarrow$ $\theta = 30°$

Hence, the angle between $\vec{a}$ and $\vec{b}$ is 30°.

➠ Short Answer Type Questions

Q. 1. If $\vec{a}$ and $\vec{b}$ are pependicular vectors, $|\vec{a}+\vec{b}| = 13$ and $|\vec{a}| = 5$, find the value of $|\vec{b}|$. (*AI, CBSE*, 2014)

Solution :

$$|\vec{a}+\vec{b}|^2 = |\vec{a}|^2+|\vec{b}|^2+2\,\vec{a}.\vec{b}$$
$\Rightarrow$ $(13)^2 = (5)^2+|\vec{b}|^2+2.(0)$ $\quad |\because \vec{a}\perp\vec{b}|$

$\Rightarrow$ $169 = 25+|\vec{b}|^2$

$\Rightarrow$ $|\vec{b}|^2 = 144$

$\Rightarrow$ $|\vec{b}| = 12$

Q. 2. If $|\vec{a}| = 8$, $|\vec{b}| = 3$ and $|\vec{a}\times\vec{b}| = 12$, find the angle between $\vec{a}$ and $\vec{b}$. [*AI CBSE*, 2014, *comptt.*]

Solution :

$$|\vec{a}\times\vec{b}| = |\vec{a}|\,|\vec{b}|\sin\theta$$
$\Rightarrow$ $12 = 8\times 3\times\sin\theta$

$\Rightarrow$ $\sin\theta = \frac{1}{2} = \sin\frac{\pi}{6}$

$\Rightarrow$ $\theta = \frac{\pi}{6}$

Q. 3. L and M are two points with position vectors $\vec{a}-\vec{b}$ and $\vec{a}+2\vec{b}$ respectively. Write the position ector of a point which divides the line segment LM n the ratio 2 : 1 externally. *(AI CBSE, 2013)*

olution :

Required position vector

$$= \frac{2(\vec{a}+2\vec{b})-1(2\vec{a}-\vec{b})}{2-1}$$

$$= 5\vec{b}$$

Q. 4. A and B are two points with position vectors $\vec{a}-3\vec{b}$ and $6\vec{b}-\vec{a}$ respectively. Write the position ector of a point P which divides the line segment B internally in the ratio 1 : 2.

(AI CBSE, 2013)

olution :

Required position vector

$$= \frac{1(0\vec{b}-\vec{a})+2(2\vec{a}-3\vec{b})}{1+2}$$

$$= \frac{3\vec{a}}{3}$$

$$= \vec{a}$$

Q. 5. Find the angle betweem x-axis amd the ector $\hat{i}+\hat{j}+\hat{k}$. *[AI CBSE, 2014 (comptt.)]*

olution :

$$(\hat{i}+\hat{j}+\hat{k}) = \hat{i}.\hat{i}+\hat{i}.\hat{j}+\hat{i}.\hat{k}$$
$$= 1+0+0$$
$$= 1$$

$$\Rightarrow |\hat{i}||\hat{i}+\hat{j}+\hat{k}| \cos\theta = 1$$

$$\Rightarrow 1.\sqrt{3}.\cos\theta = 1$$

$$\Rightarrow \cos\theta = \frac{1}{\sqrt{3}}$$

$$\Rightarrow \theta = \cos^{-1}\left(\frac{1}{\sqrt{3}}\right).$$

Q. 6. Write the value of curve of the angle which he vector $\vec{a}=\vec{i}+\vec{j}+\vec{k}$ makes with y-axis.

[CBSE, 2014 (Comptt.)]

olution :

$$\hat{j}\cdot(\hat{i}+\hat{j}+\hat{k}) = \hat{j}.\hat{i}+\hat{j}.\hat{j}+\hat{j}.\hat{k}$$
$$= 0+1+0$$
$$= 1$$

$$\Rightarrow |\hat{j}||\hat{i}+\hat{j}+\hat{k}| \cos\theta = 1$$

$$\Rightarrow 1.\sqrt{3}.\cos\theta = 1$$

$$\Rightarrow \cos\theta = \frac{1}{\sqrt{3}}$$

Q. 7. Find the projection of the vector $\vec{i}+3\vec{j}+7\vec{k}$ on the vector $2\vec{i}-3\vec{j}+6\vec{k}$.

(CBSE, 2014)

Solution :

Required projection

$$= \frac{(\hat{i}+3\hat{j}+7\hat{k}).(2\hat{i}-3\hat{j}+6\hat{k})}{|2\hat{i}-3\hat{j}+6\hat{k}|}$$

$$= \frac{(1)(2)+(3)(-3)+(7)(6)}{\sqrt{(2)^2+(-3)^2+(6)^2}}$$

$$= \frac{2-9+42}{7}$$

$$= \frac{35}{7}$$

$$= 5$$

Q. 8. P and Q are two points with position vectors $3\vec{a}-2\vec{b}$ and $\vec{a}+\vec{b}$ respectively. Write the position vector of a point R which divides the line seement PQ in the ratio 2 : 1 externally.

(CBSE, 2014)

Solution :

Required projection

$$= \frac{2(\vec{a}+\vec{b})-1(3\vec{a}-2\vec{b})}{2-1}$$

$$= -\vec{a}+4\vec{b}$$

Q. 9. Find a vector $\vec{a}$ of magnitude $5\sqrt{2}$, making an angle of $\frac{\pi}{4}$ with x-axis, $\frac{\pi}{2}$ with y-axis and an acute angle θ with z-axis. *(AI CBSE, 2014)*

Solution :

$$l^2+m^2+n^2 = 1$$

$$\Rightarrow \cos^2\frac{\pi}{4}+\cos^2\frac{\pi}{2}+\cos^2\theta = 1$$

$$\Rightarrow \frac{1}{2}+0+\cos^2\theta = 1$$

$$\Rightarrow \cos\theta = \frac{1}{\sqrt{2}} = \cos\frac{\pi}{4}$$

$$\Rightarrow \theta = \frac{\pi}{4}$$

⇒ Unit vector

$$= \frac{1}{\sqrt{2}}\hat{i}+0\hat{j}+\frac{1}{\sqrt{2}}\hat{k}$$

$$= \frac{1}{\sqrt{2}}\hat{i}+\frac{1}{\sqrt{2}}\hat{k}$$

∴ Required vector

$$= 5\sqrt{2}\left\{\frac{1}{\sqrt{2}}\hat{i}+\frac{1}{\sqrt{2}}\hat{k}\right\}$$

$$= 5\hat{i}+5\hat{k}$$

Q. 10. Write a vector in the direction of the vector $\hat{i}+2\hat{j}+2\hat{k}$ that has magnitude a units.

(*CBSE*, 2014 (*Comptt.*)

Solution :

Unit vector in the direction of the vector $\hat{i}+2\hat{j}+2\hat{k}$

$$= \frac{\hat{i}+2\hat{j}+2\hat{k}}{|\hat{i}+2\hat{j}+2\hat{k}|}$$

$$= \frac{\hat{i}+2\hat{j}+2\hat{k}}{\sqrt{(1)^2+(2)^2+(2)^2}}$$

$$= \frac{\hat{i}-2\hat{j}+2\hat{k}}{3}$$

$\therefore$ Required vector

$$= 9\,\frac{\hat{i}+2\hat{j}+2\hat{k}}{3}$$

$$= 3(\hat{i}+2\hat{j}+2\hat{k})$$

Q. 11. Find a unit vector perpendicular to each of the vectors $(\vec{a}+\vec{b})$ and $(\vec{a}-\vec{b})$, where $\vec{a} = \hat{i}+\hat{j}+\hat{k}$, $\vec{b} = \hat{i}+2\hat{j}+3\hat{k}$. (*USEB*, 2014)

Solution :

$$\vec{a}+\vec{b} = 2\hat{i}+3\hat{j}+4\hat{k}$$

$$\vec{a}-\vec{b} = -\hat{j}-2\hat{k}$$

$$\therefore \quad (\vec{a}+\vec{b})\times(\vec{a}-\vec{b}) = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & 3 & 4 \\ 0 & -1 & -2 \end{vmatrix}$$

$$= -2\hat{i}+4\hat{j}-2\hat{k}$$

Also, $|-2\hat{i}+4\hat{j}-2\hat{k}| = \sqrt{(-2)^2+(4)^2+(-2)^2}$

$$= \sqrt{4+16+4}$$

$$= 2\sqrt{6}$$

$\therefore$ Required unit vector

$$= \frac{-2\hat{i}+4\hat{j}-2\hat{k}}{2\sqrt{6}}$$

$$= -\frac{1}{\sqrt{6}}\hat{i}+\frac{2}{\sqrt{6}}\hat{j}-\frac{1}{\sqrt{6}}\hat{k}$$

Q. 12. If $\hat{i}+\hat{j}+\hat{k}$, $2\hat{i}+5\hat{j}$, $3\hat{i}+2\hat{j}-3\hat{k}$ and $\hat{i}-6\hat{j}-\hat{k}$ are the position vectors of points A, B, C and D respectively then find the angle between $\overrightarrow{AB}$ and $\overrightarrow{CD}$. (*BSER*, 2014)

Solution :

$$\overrightarrow{AB} = (2\hat{i}+5\hat{j})-(\hat{i}+\hat{j}+\hat{k})$$

$$= \hat{i}+4\hat{j}-\hat{k}$$

and $\overrightarrow{CD} = (\hat{i}+6\hat{j}+\hat{k})-(3\hat{i}+2\hat{j}-3\hat{k})$

$$= -2\hat{i}-8\hat{j}+2\hat{k}$$

$$\therefore \quad \overrightarrow{AB}.\overrightarrow{CD} = (1)(-2)+(4)(-8)+(-1)(2)$$

$$= -2-32-2 = -36$$

$$\overrightarrow{AB} = \sqrt{(1)^2+(4)^2+(-1)^2} = \sqrt{18}$$

$$\overrightarrow{CD} = \sqrt{(-2)^2+(-8)^2+(2)^2} = \sqrt{72}$$

$$\overrightarrow{AB}.\overrightarrow{CD} = |\overrightarrow{AB}||\overrightarrow{CD}|\cos\theta$$

$$\Rightarrow \quad -36 = \sqrt{18}\sqrt{72}\cos\theta$$

$$\Rightarrow \quad -36 = 3\sqrt{2}\,6\,\sqrt{2}\cos\theta$$

$$\Rightarrow \quad \cos\theta = -1 = \cos 180^\circ$$

$$\Rightarrow \quad \theta = 180^\circ$$

Hence, the angle between $\overrightarrow{AB}$ and $\overrightarrow{CD}$ is 180°.

Q. 13. If $\vec{a} = 2\hat{i}-3\hat{j}+\hat{k}$, $\vec{b} = \hat{i}+\hat{k}$, $\vec{c} = 2\hat{j}-\hat{k}$ are three vectors, find the area of the parallelogram having diagonals $(\vec{a}+\vec{b})$ and $(\vec{b}+\vec{c})$.

[*CBSE*, 2014; (*Comptt.*)]

Solution :

$$\vec{a}+\vec{b} = \hat{i}-3\hat{j}+2\hat{k}$$

and $\vec{b}+\vec{c} = -\hat{i}+2\hat{j}$

$$(\vec{a}+\vec{b})\times(\vec{b}+\vec{c}) = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & -3 & 2 \\ -1 & 2 & 0 \end{vmatrix}$$

$$= -4\hat{i}-2\hat{j}-\hat{k}$$

$$\therefore \quad |(\vec{a}+\vec{b})\times(\vec{b}+\vec{c})| = \sqrt{(-4)^2+(-2)^2+(-1)^2}$$

$$= \sqrt{16+4+1} = \sqrt{21}$$

$\therefore$ Area of the parallelogram

$$= \frac{1}{2}\sqrt{21} \text{ square units}$$

Q. 14. Write the projection of $\vec{b}+\vec{c}$, on $\vec{a}$, where $\vec{a} = 2\hat{i}-2\hat{j}+\hat{k}$, $\vec{b} = \hat{i}+2\hat{j}-2\hat{k}$ and $\vec{c} = 2\hat{i}-\hat{j}+4\hat{k}$.

[*CBSE*, 2013, (*Comptt.*)]

Solution :

$$\vec{b}+\vec{c} = (\hat{i}+2\hat{j}-2\hat{k})+(2\hat{i}-\hat{j}+4\hat{k})$$

$$= 3\hat{i} + \hat{j} + 2\hat{k}$$

and $$\vec{a} = 2\hat{i} - 2\hat{j} + \hat{k}$$

projection of $\vec{b} + \vec{c}$ on $\vec{a}$

$$= \frac{(\vec{b} + \vec{c}).\vec{a}}{|\vec{a}|}$$

$$= \frac{(3\hat{i} + \hat{j} + 2\hat{k}).(2\hat{i} - 2\hat{j} + \hat{k})}{|2\hat{i} - 2\hat{i} + \hat{k}|}$$

$$= \frac{(3)(2) + (1)(-2) + (2)(1)}{\sqrt{(2)^2 + (-2)^2 + (1)^2}}$$

$$= \frac{6 - 2 + 2}{3}$$

$$= 2$$

Q. 15. If $\vec{a} = \hat{i} - \hat{j} + 7\hat{k}$ and $\vec{b} = 5\hat{i} - \hat{j} + \lambda\hat{k}$, then find the value of λ, so that $\vec{a} + \vec{b}$ and $\vec{a} - \vec{b}$ are perpendicular vectors. *(AI CBSE, 2013)*

Solution :

$$\vec{a} + \vec{b} = 6\hat{i} - 2\hat{j} + (7 + \lambda)\hat{k}$$

$$\vec{a} - \vec{b} = -4\hat{i} + (7 - \lambda)\hat{k}$$

If $\vec{a} + \vec{b}$ and $\vec{a} - \vec{b}$ are perpendicular vectors, then

$$(\vec{a} + \vec{b}).(\vec{a} - \vec{b}) = 0$$

$$\Rightarrow \{6\hat{i} - 2\hat{j} + (7 + \lambda)\hat{k}\}.\{-4\hat{j} + (7 + \lambda)\hat{k}\} = 0$$

$$\Rightarrow -24 + (7 + \lambda)(7 - \lambda) = 0$$

$$\Rightarrow -24 + 4a - \lambda^2 = 0$$

$$\Rightarrow 25 - \lambda^2 = 0$$

$$\Rightarrow \lambda^2 = 25$$

$$\therefore \lambda = \pm 5$$

Q. 16. If $\vec{a}$ and $\vec{b}$ are two unit vectors such that $\vec{a} + \vec{b}$ is also a unit vector, then find the angle between $\vec{a}$ and $\vec{b}$. *(CBSE, 2014)*

Solution :

$$|\vec{a}| = 1$$

$$|\vec{b}| = 1$$

and $$|\vec{a} + \vec{b}| = 1$$

Now,

$$(\vec{a} + \vec{b})(\vec{a} + \vec{b}) = |\vec{a} + \vec{b}|^2$$

$$\Rightarrow \vec{a}.\vec{a} + \vec{a}.\vec{b} + \vec{b}.\vec{a} + \vec{b}.\vec{b} = 1^2$$

$$\Rightarrow \vec{a}^2 + 2\vec{a}.\vec{b} + \vec{b}^2 = 1^2$$

$$\Rightarrow 1^2 + 2.1.1.\cos\theta + 1^2 = 1^2$$

$$\Rightarrow 1 + 2\cos\theta + 1 = 1$$

$$\Rightarrow 2\cos\theta + 1 = 1$$

$$\Rightarrow 2\cos\theta = -1$$

$$\Rightarrow \cos\theta = -\frac{1}{2} = \cos\frac{2\pi}{3}$$

$$\Rightarrow \theta = \frac{2\pi}{3}$$

Hence, the angle between $\vec{a}$ and $\vec{b}$ is $\frac{2\pi}{3}$.

Q. 17. The two adjacent sides of a parallelogram are $2\hat{i} - 4\hat{j} + 5\hat{k}$ and $\hat{i} - 2\hat{j} - 3\hat{k}$. Find the unit vector parallel to its diagonal. Also, find its area. *(USEB, 2013)*

Solution :

$$(2\hat{i} - 4\hat{j} + 5\hat{k}) \times (\hat{i} - 2\hat{j} - 3\hat{k})$$

$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & -4 & 5 \\ 1 & -2 & -3 \end{vmatrix}$$

$$= (12 + 10)\hat{i} + (5 + 6)\hat{j} + (-4 + 4)\hat{k}$$

$$= 22\hat{i} + 11\hat{j}$$

$$|22\hat{i} + 11\hat{j}| = \sqrt{(22)^2 + (11)^2}$$

$$= \sqrt{484 + 121}$$

$$= \sqrt{605} = 11\sqrt{5}$$

$$\therefore \text{Required unit vector} = \frac{22\hat{i} + 11\hat{j}}{11\sqrt{5}}$$

$$= \frac{2\hat{i} + \hat{j}}{\sqrt{5}}$$

$$= \frac{2}{\sqrt{5}}\hat{i} + \frac{1}{\sqrt{5}}\hat{j}$$

Also, area of the parallelogram

$$= 11\sqrt{5}$$

Q. 18. For what value of λ and vector $\hat{i} + 2\lambda\hat{j} + \hat{k}$ and $2\hat{i} + \hat{j} - 3\hat{k}$ are perpendicular ? *(CBSE, 2014)*

Solution :

$$(\hat{i} + 2\lambda\hat{j} + \hat{k}).(2\hat{i} + \hat{j} - 3\hat{k}) = 0$$

$$\Rightarrow 2 + 2\lambda - 3 = 1$$

$$\Rightarrow 2\lambda = 1$$

$$\therefore \lambda = \frac{1}{2}$$

Q. 19. Write a unit vector in the direction of the sum of the vectors $\vec{a} = 2\hat{i} + 2\hat{j} - 5\hat{k}$ and $\vec{b} = 2\hat{i} + \hat{j} - 7\hat{k}$.

Solution : (*CBSE*, 2014)

$$\vec{a}+\vec{b} = 4\hat{i}+3\hat{j}-12\hat{k}$$

$$\Rightarrow \quad |\vec{a}+\vec{b}| = \sqrt{(4)^2+(3)^2+(-12)^2}$$

$$= \sqrt{16+9+144}$$

$$= \sqrt{169}$$

$$= 13$$

∴ Required unit vector

$$= \frac{\vec{a}\times\vec{b}}{|\vec{a}\times\vec{b}|}$$

$$= \frac{4\hat{i}+3\hat{j}-12\hat{k}}{13}$$

$$= \frac{4}{13}\hat{i}+\frac{3}{13}\hat{j}-\frac{12}{13}\hat{k}$$

Q. 20. If a unit vector $\vec{a}$ makes angles $\frac{\pi}{3}$ with [î, an angle] $\frac{\pi}{4}$ with $\hat{j}$ and an acute angle θ with $\hat{k}$, then find [th]e value of θ.

[S]olution : (*CBSE*, 2013)

$$l^2+m^2+n^2 = 1$$

$$\Rightarrow \cos^2\frac{\pi}{3}+\cos^2\frac{\pi}{4}+\cos^2\theta = 1$$

$$\Rightarrow \left(\frac{1}{2}\right)^2+\left(\frac{1}{\sqrt{2}}\right)^2+\cos^2\theta = 1$$

$$\Rightarrow \frac{1}{4}+\frac{1}{2}+\cos^2\theta = 1$$

$$\Rightarrow \cos^2\theta = \frac{1}{4}$$

$$\Rightarrow \cos\theta = \frac{1}{2} = \cos\frac{\pi}{3}$$

$$\theta = \frac{\pi}{3}$$

[Q.] 21. Using vectors, find the area of the triangle [w]ith vectors A (1, 2, 3), B (2, –1, 4) and C (4, 5, –1).

(*CBSE*, 2013; *AI CBSE*, 13)

[Soluti]on :

[Let] O be the origin of vectors, then

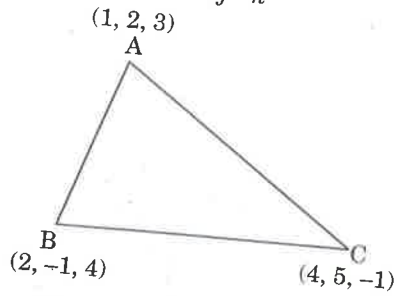
$$\vec{OA} = \hat{i}+2\hat{j}+3\hat{k}$$

$$\vec{OB} = 2\hat{i}-\hat{j}+4\hat{k}$$

$$\vec{OC} = 4\hat{i}+5\hat{j}-\hat{k}$$

$$\vec{BA} = \vec{OA}-\vec{OB}$$

$$= -\hat{i}+3\hat{j}-\hat{k}$$



$$\vec{BC} = 2\hat{i}+6\hat{j}-5\hat{k}$$

$$\therefore \quad \vec{BA}\times\vec{BC} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ -1 & 3 & -1 \\ 2 & 6 & -5 \end{vmatrix}$$

$$= (-15+6)\hat{i}+(-2-5)\hat{i}+(-6-6)\hat{k}$$

$$= -9\hat{i}-7\hat{j}-12\hat{k}$$

$$\therefore \quad |\vec{BA}\times\vec{BC}| = \sqrt{(-9)^2+(-7)^2+(-12)^2}$$

$$= \sqrt{81+49+144}$$

$$= \sqrt{274}$$

∴ Area of ΔABC $= \frac{1}{2}|\vec{BA}\times\vec{BC}| = \frac{1}{2}\sqrt{274}$ square units

Q. 22. The angle between two unit vectors $\hat{a}$ and $\hat{b}$ is θ. Prove that

$$\sin\frac{\theta}{2} = \frac{1}{2}|\hat{a}-\hat{b}|.$$

Solution : (*BSER*, 2013)

$$(\hat{a}-\hat{b}).(\hat{a}-\hat{b}) = |\hat{a}-\hat{b}|^2$$

$$\Rightarrow \hat{a}.\hat{a}-\hat{a}.\hat{b}.-\hat{b}.\hat{a}.+\hat{b}.\hat{b} = |\hat{a}-\hat{b}|^2$$

$$\Rightarrow |\hat{a}|^2-2\hat{a}.\hat{b}+|\hat{b}|^2 = |\hat{a}-\hat{b}|^2$$

$$\Rightarrow 1^2-2.1.1.\cos\theta+1^2 = |\hat{a}-\hat{b}|^2$$

$$\Rightarrow 2(1-\cos\theta) = |\hat{a}-\hat{b}|^2$$

$$\Rightarrow 2.2\sin^2\frac{\theta}{2} = |\hat{a}-\hat{b}|^2$$

$$\Rightarrow 4\sin^2\frac{\theta}{2} = |\hat{a}-\hat{b}|^2$$

$$\Rightarrow 2\sin\frac{\theta}{2} = |\hat{a}-\hat{b}|$$

$$\Rightarrow \sin\frac{\theta}{2} = \frac{1}{2}|\hat{a}-\hat{b}|$$

Q. 23. If $\vec{a}+\vec{b}+\vec{c} = \vec{u}$, show that $\vec{a}\times\vec{b} = \vec{b}\times\vec{c} = \vec{c}\times\vec{a}$.

Solution : (*BSEB*, 2014)

$$\vec{a}+\vec{b}+\vec{c} = \vec{0}$$

$$\Rightarrow \quad \vec{a}+\vec{b} = -\vec{c}$$

$$\Rightarrow \quad (\vec{a}+\vec{b})\times\vec{b} = -\vec{c}\times\vec{b}$$

$$\Rightarrow \quad \vec{a}\times\vec{b}+\vec{b}\times\vec{b} = \vec{b}\times\vec{c}$$

$$\Rightarrow \quad \vec{a}\times\vec{b}+\vec{0} = \vec{b}\times\vec{c}$$

$$\Rightarrow \quad \vec{a}\times\vec{b} = \vec{b}\times\vec{c} \qquad ...(1)$$

Again, $\vec{a}+\vec{b}+\vec{c} = \vec{0}$

$$\Rightarrow \quad \vec{b}+\vec{c} = -\vec{a}$$

$$\Rightarrow \quad (\vec{b}+\vec{c})\times\vec{c} = -\vec{a}\times\vec{c}$$

$$\Rightarrow \quad \vec{b}\times\vec{c}+\vec{c}\times\vec{c} = \vec{c}\times\vec{a}$$

$$\Rightarrow \quad \vec{b}\times\vec{c}+\vec{0} = \vec{c}\times\vec{a}$$

$$\Rightarrow \quad \vec{b}\times\vec{c} = \vec{c}\times\vec{a} \qquad ..(2)$$

From (1) and (2), we get

$$\vec{a}\times\vec{b} = \vec{b}\times\vec{c} = \vec{c}\times\vec{a}$$

Q. 24. Prove that $|\vec{a}\times\vec{b}|^2 = \begin{vmatrix} \vec{a}.\vec{a} & \vec{a}.\vec{b} \\ \vec{a}.\vec{b} & \vec{b}.\vec{b} \end{vmatrix}$

(*BSEB*, 2013)

Solution :

$$\text{LHS} = |\vec{a}\times\vec{b}|^2$$

$$= |ab\sin\theta\,\hat{n}|^2$$

$$= a^2b^2\sin^2\theta \qquad ...(1)$$

$$\text{RHS} = \begin{vmatrix} \vec{a}.\vec{a} & \vec{a}.\vec{b} \\ \vec{a}.\vec{b} & \vec{b}.\vec{b} \end{vmatrix}$$

$$= (\vec{a}.\vec{a})(\vec{b}.\vec{b})-(\vec{a}.\vec{b})(\vec{a}.\vec{b})$$

$$= a^2b^2-(\vec{a}.\vec{b})^2$$

$$= a^2b^2-(ab\cos\theta)^2$$

$$= a^2b^2-a^2b^2\cos^2\theta$$

$$= a^2b^2(1-\cos^2\theta)$$

$$= a^2b^2\sin^2\theta \qquad ...(2)$$

From (1) and (2),

LHS = RHS

Q. 25. If $\vec{a} = \vec{i}+\vec{j}+\vec{k}$ and $\vec{b} = \vec{j}-\vec{k}$, find a vector $\vec{c}$, such that $\vec{a}\times\vec{c} = \vec{b}$ and $\vec{a}.\vec{c} = 3$. (*CBSE*, 2013)

Solution :

Let $\vec{c} = c_1\hat{i}+c_2\hat{j}+c_3\hat{k}$

$$\vec{a}\times\vec{c} = \vec{b}$$

$$\Rightarrow \quad \begin{vmatrix} i & j & k \\ 1 & 1 & 1 \\ c_1 & c_2 & c_3 \end{vmatrix} = \vec{j}-\vec{k}$$

$$\Rightarrow (c_3-c_2)\,\hat{i}+(c_1-c_3)\,\hat{j}+(c_2-c_1)\,\vec{k} = \vec{j}-\vec{k}$$

Comparing the cofficients, we get

$$c_3-c_2 = 0 \qquad ...(1)$$

$$c_1-c_3 = 0 \qquad ...(2)$$

$$c_2-c_1 = 0 \qquad ...(3)$$

Also, $\vec{a}.\vec{c} = 3$

$$\Rightarrow \quad c_1+c_2+c_3 = 3 \qquad ...(4)$$

$$\Rightarrow \quad c_1+2c_2 = 3 \qquad ...(5)\ \text{[Using (1)]}$$

Saving (3) and (5), we get

$$c_2 = \frac{2}{3},\ c_1 = \frac{5}{3}$$

$$\therefore \quad c_3 = \frac{2}{3}$$

$$\therefore \quad \vec{c} = \frac{5}{3}\hat{i}+\frac{2}{3}\hat{j}+\frac{2}{3}\hat{k}$$

Long Answer Type Questions

Q. 1. Find the values of λ for which the angle between the vectors $\vec{a} = 2\lambda^2\hat{i}+4\lambda\hat{j}+\hat{k}$ and $\vec{b} = 7\hat{i}-2\hat{j}+\lambda\hat{k}$ is obtuse. [*CBSE*, 2013 (*Comptt.*)]

Solution :

$$\vec{a}.\vec{b} = ab\cos\theta$$

$$\Rightarrow \quad \cos\theta = \frac{\vec{a}.\vec{b}}{ab}$$

$$= \frac{(2\lambda^2\hat{i}+4\lambda\hat{j}+\hat{k}).(7\hat{i}-2\hat{j}+\lambda\hat{k})}{|7\lambda^2\hat{i}+4\lambda\hat{j}+\hat{k}|\,|7\hat{i}-2\hat{j}+\lambda\hat{k}|}$$

$$= \frac{(2\lambda^2).(7)+(4\lambda)(-2)+(1)(\lambda)}{\sqrt{4\lambda^4+16\lambda^2+1}\,\sqrt{49+4+\lambda^2}}$$

$$= \frac{14\lambda^2-8\lambda+\lambda}{\sqrt{4\lambda^4+16\lambda^2+1}\,\sqrt{\lambda^2+53}}$$

$$= \frac{14\lambda^2-7\lambda}{\sqrt{4\lambda^4+16\lambda^2+1}\,\sqrt{\lambda^2+53}}$$

If θ is obtuse, then

$$\cos\theta < 0$$

$$\Rightarrow \quad 14\lambda^2-7 < 0$$

$$\Rightarrow \quad 7\lambda(2\lambda-1) < 0$$

$$\Rightarrow \quad \lambda(2\lambda-1) < 0$$

$$\Rightarrow \quad 2\lambda\left(\lambda-\frac{1}{2}\right) < 0$$

$$\Rightarrow \quad \lambda\left(\lambda-\frac{1}{2}\right) < 0$$

$$\Rightarrow \quad 0 < \lambda < \frac{1}{2}$$

Q. 2. If $\vec{a} = \vec{i} + 4\vec{j} + 2\vec{k}$, $\vec{b} = 3\vec{i} - 2\vec{j} + 7\vec{k}$ and $\vec{c} = 2\vec{i} - \vec{j} + 4\vec{k}$, then find a vector $\vec{d}$ which is perpendicular to $\vec{a}$ and $\vec{b}$ and $\vec{c}\,\vec{d} = 15$. *(BSER, 2014)*

Solution :

Let $\vec{d} = d_1\hat{i} + d_2\hat{j} + d_3\hat{k}$

$\therefore \quad \vec{d} = \vec{a}$

$\vec{d}\cdot\vec{a} = 0$

$\Rightarrow \quad d_1 + 4d_2 + 2d_3 = 0 \quad$ ...(1)

$\therefore \quad \vec{d} = \vec{b}$

$\therefore \quad \vec{d}\cdot\vec{b} = 0$

$\Rightarrow \quad 3d_1 - 2d_2 + 7d_3 = 0 \quad$ ...(2)

From (1) and (2),

$$\frac{d_1}{28+4} = \frac{d_2}{6-7} = \frac{d_3}{-2-12}$$

$\Rightarrow \quad \dfrac{d_1}{32} = \dfrac{d_2}{-1} = \dfrac{d_3}{-14} = \lambda(\text{say})$

$\Rightarrow \quad d_1 = 32\lambda$

$d_2 = -\lambda$

$d_3 = -14\lambda$

Now, $\vec{c}.\vec{d} = 15$

$\Rightarrow \quad 2d_1 - d_2 + 4d_3 = 15$

$\Rightarrow \quad 64\lambda + \lambda - 56\lambda = 15$

$\Rightarrow \quad 9\lambda = 15$

$\Rightarrow \quad \lambda = \dfrac{5}{3}$

$\therefore \quad d_1 = \dfrac{160}{3}$,

$d_2 = -\dfrac{5}{3}$

and $\quad d_3 = -\dfrac{70}{3}$

$\therefore \quad \vec{d} = \dfrac{160}{3}\hat{i} - \dfrac{5}{3}\hat{j} - \dfrac{70}{3}\hat{k}$

Q. 3. $\vec{a}, \vec{b}$ are $\vec{c}$ are three vectors such that $|\vec{a}| = 3, |\vec{b}| = 4, |\vec{c}| = 5$ and each one of them is perpendicular to the sum of the other two, find $|\vec{a} + \vec{b} + \vec{c}|$. *(USEB, 2013)*

Solution :

$\vec{a} \perp (\vec{b} + \vec{c})$

$\Rightarrow \quad \vec{a}\cdot(\vec{b} + \vec{c}) = 0$

$\Rightarrow \quad \vec{a}\cdot\vec{b} + \vec{a}\cdot\vec{c} = 0 \quad$ ...(1)

Similarly, $\quad \vec{b} \perp (\vec{c} + \vec{a})$

$\Rightarrow \quad \vec{b}\cdot(\vec{c} + \vec{a}) = 0$

$\Rightarrow \quad \vec{b}\cdot\vec{c} + \vec{b}\cdot\vec{a} = 0 \quad$ ...(2)

and $\quad \vec{c} \perp (\vec{a} + \vec{b})$

$\Rightarrow \quad \vec{c}\cdot(\vec{a} + \vec{b}) = 0$

$\Rightarrow \quad \vec{c}\cdot\vec{a} + \vec{c}\cdot\vec{b} = 0$

$\Rightarrow \quad \vec{c}\cdot\vec{a} + \vec{b}\cdot\vec{c} = 0 \quad$ ...(3)

Adding (1), (2) and (3), we get

$2(\vec{a}.\vec{b} + \vec{b}.\vec{c}. + \vec{c}.\vec{a}) = 0$

$\Rightarrow \quad \vec{a}.\vec{b} + \vec{b}.\vec{c}. + \vec{c}.\vec{a} = 0 \quad$ ...(4)

Now, $\quad |\vec{a} + \vec{b} + \vec{c}|^2 = (\vec{a} + \vec{b} + \vec{c})\,(\vec{a} + \vec{b} + \vec{c})$

$= \vec{a}^2 + \vec{b}^2 + \vec{c}^2 + 2(\vec{a}.\vec{b}. + \vec{b}.\vec{c} + \vec{c}.\vec{a})$

$= 9 + 16 + 25 + 2(0)$

$= 50$

$\Rightarrow \quad |\vec{a} + \vec{b} + \vec{c}| = 5\sqrt{2}$

Q. 4. Find the vector $\vec{p}$ which is perpendicular to both $\vec{\alpha} = 4\hat{i} + 5\hat{j} - \hat{k}$ and $\vec{\beta} = \hat{i} + 4\hat{j} + 5\hat{k}$ and $\vec{p}.\vec{q} = 21$, where $\vec{q} = 3\hat{i} + \hat{j} - \hat{k}$. *[AI CBSE, 2014 (Comptt.)]*

Solution :

Let $\quad \vec{p} = p_1\hat{i} + p_2\hat{j} + p_3\hat{k}$

$\because \quad \vec{p} \perp \vec{\alpha}$

$\therefore \quad \vec{p}\cdot\vec{\alpha} = 0$

$\Rightarrow \quad 4p_1 + 5p_2 - p_3 = 0 \quad$ ...(1)

$\because \quad \vec{p} \perp \vec{\beta} = 0$

$\therefore \quad \vec{p}.\vec{\beta} = 0$

$\Rightarrow \quad p_1 - 4p_2 + 5p_3 = 0 \quad$ ...(2)

From (1) and (2),

$$\frac{p_1}{25-4} = \frac{p_2}{-1-20} = \frac{p_3}{-16-5}$$

$\Rightarrow \quad \dfrac{p_1}{21} = \dfrac{p_2}{-21} = \dfrac{p_3}{-21}$

$\Rightarrow \quad \dfrac{p_1}{1} = \dfrac{p_2}{-1} = \dfrac{p_3}{-1} = \lambda\,(\text{say})$

$\Rightarrow \quad p_1 = \lambda$

$p_2 = -\lambda$

and $\quad p_3 = -\lambda$

Now $\quad \vec{p}.\vec{q} = 21$

$\Rightarrow \quad 3p_1 + p_2 - p_3 = 21$

$\Rightarrow \quad 3\lambda - \lambda + \lambda = 21$

$\Rightarrow \quad 3\lambda = 21$

$\Rightarrow \quad \lambda = 7$

$\therefore \quad \vec{p} = 7\hat{i} - 7\hat{j} - 7\hat{k}$

Q. 5. The scalar product of the vector $\vec{a} = \hat{i} + \hat{j} + \hat{k}$ with a unit vector along the sum of vectors $\vec{b} = 2\hat{i} + 4\hat{j} - 5\hat{k}$ and $\vec{c} = \lambda\hat{i} + 2\hat{j} + 3\hat{k}$ is equal to one. Find the value of λ and hence find the unit vector along $\vec{b} + \vec{c}$. (*At CBSE*, 2014)

Solution :

Let $\vec{a} = \hat{i} + \hat{j} + \hat{k}$

$\vec{b} + \vec{c} = (2 + \lambda)\hat{i} + 6\hat{j} - 2\hat{k}$

Unit vector along $\vec{b} + \vec{c}$

$$= \frac{(2+\lambda)\hat{i} + 6\hat{j} - 2\hat{k}}{\sqrt{(2+\lambda)^2 + (6)^2 + (-2)^2}}$$

$$= \frac{(2+\lambda)\hat{i} + 6\hat{j} - 2\hat{k}}{\sqrt{\lambda^2 + 4\lambda + 44}}$$

According to the question,

$$(\hat{i} + \hat{j} + \hat{k}) \cdot \left[\frac{(2+\lambda)\hat{i} + 6\hat{j} - 2\hat{k}}{\sqrt{(\lambda^2 + 4\lambda + 44)}}\right] = 1$$

$\Rightarrow \quad 2 + \lambda + b - 2 = \sqrt{\lambda^2 + 4\lambda + 44}$

$\Rightarrow \quad \lambda + \beta = \sqrt{\lambda^2 + 4\lambda + 44}$

$\Rightarrow \quad (\lambda + \beta)^2 = \lambda^2 + 4\lambda + 44$

$\Rightarrow \quad \lambda^2 + 12\lambda + 36 = \lambda^2 + 4\lambda + 44$

$\Rightarrow \quad 8\lambda = 8$

$\therefore \quad \lambda = 1$

Hence the value of λ is 1.

$\therefore$ Unit vector along $\vec{b} + \vec{c}$

$$= \frac{(2+1)\hat{i} + 6\hat{j} - 2\hat{k}}{\sqrt{(1)^2 + 4(1) + 44}}$$

$$= \frac{3\hat{i} + 6\hat{j} - 2\hat{k}}{7}$$

$$= \frac{3}{7}\hat{i} + \frac{6}{7}\hat{j} - \frac{2}{7}\hat{k}$$

Q. 6. Find $\vec{a}.(\vec{b} \times \vec{c})$, if $\vec{a} = 2\hat{i} + \hat{j} + 3\hat{k}$, $\vec{b} = -\hat{i} + 2\hat{j} + \hat{k}$ and $\vec{c} = 3\hat{i} + \hat{j} + 2\hat{k}$. (*AI CBSE*, 2014)

Solution :

Let $\vec{a}.(\vec{b} \times \vec{c}) = \begin{vmatrix} 2 & 1 & 3 \\ -1 & 2 & 1 \\ 3 & 1 & 2 \end{vmatrix}$

$= 2(4-1) + 1(3+2) + 3(-1-6)$

$= 6 + 5 - 21$

$= -10$

Q. 7. If $\vec{a_1} = \hat{i} + 2\hat{j} + \hat{k}$, $\vec{b_1} = \hat{i} - \hat{j} + \hat{k}$, $\vec{a_2} = 2\hat{i} - \hat{j} + \hat{k}$ and $\vec{b_2} = 2\hat{i} + \hat{j} + 2\hat{k}$, then find : (*JAC.*, 2014)

(i) $\vec{a_2} - \vec{a_1}$ (ii) $\vec{b_2} - \vec{b_1}$

(iii) $\vec{b_1} \times \vec{b_2}$ (iv) $\vec{a_1} \times \vec{a_2}$

(v) $\vec{b_1} \times \vec{b_2} \cdot (\vec{a_2} - \vec{a_1})$

Solution :

(i) $\vec{a_2} - \vec{a_1} = (2\hat{i} - \hat{j} - \hat{k}) - (\hat{i} + 2\hat{j} + \hat{k})$

$= \hat{i} - 3\hat{j} - 2\hat{k}$

(ii) $\vec{b_2} - \vec{b_1} = (2\hat{i} + \hat{j} + 2\hat{k}) - (\hat{i} - \hat{j} + \hat{k})$

$= i + 2j + k$

(iii) $\vec{b_1} \times \vec{b_2} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & -1 & 1 \\ 2 & 1 & 2 \end{vmatrix}$

$= (-2-1)\hat{i} + (2-2)\hat{j} + (1+2)\hat{k}$

$= -3\hat{i} + 3\hat{k}$

(iv) $\vec{a_1} \times \vec{a_2} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 2 & 1 \\ 2 & -1 & -1 \end{vmatrix}$

$= (-2+1)\hat{i} + (2+1)\hat{j} + (-1-4)\hat{k}$

$= -\hat{i} + 3\hat{j} - 5\hat{k}$

(v) $(\vec{b_1} \times \vec{b_2})(\vec{a_2} - \vec{a_1})$

$= (-3\hat{i} + 3\hat{k}).(\hat{i} - 3\hat{j} - 2\hat{k})$

$= (-3)(-1) + (0)(-3) + (3)(-2)$

$= 3 + 0 - 6$

$= -3$

Q. 8. If $\vec{a} = \hat{i} - 3\hat{j} + 4\hat{k}$ and $\vec{b} = 2\hat{i} + \hat{j} + \hat{k}$, then find the following :

(i) $|\vec{a}|$ **(ii) $\vec{a}.\vec{b}$**

(iii) $\vec{a} \times \vec{b}$ (iv) projection of $\vec{a}$ on $\vec{b}$ (*JAC*, 2014)

Solution :

(i) $|\vec{a}| = |\hat{i} - 3\hat{j} + 4\hat{k}|$

$= \sqrt{(1)^2 + (-3)^2 + (4)^2}$

$= \sqrt{1 + 9 + 16}$

$= \sqrt{26}$

(ii) $\vec{a} \times \vec{b} = (\hat{i} - 3\hat{j} + 4\hat{k}).(2\hat{i} + \hat{j} + \hat{k})$

$= (1)(2) + (-3)(1) + (4)(1)$

$= 2 - 3 + 4$

$= 3$

(iii) $\vec{a} \times \vec{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & -3 & 4 \\ 2 & 1 & 1 \end{vmatrix}$

$= (-3-4)\hat{i} + (8-1)\hat{j} + (1+6)\hat{k}$

$= -7(-\hat{i} + \hat{j} + \hat{k})$

(iv) $|\vec{b}| = |2\hat{i} + \hat{j} + \hat{k}|$

$= \sqrt{(2)^2 + (1)^2 + (1)^2}$

$= \sqrt{4+1+1}$

$= \sqrt{6}$

Projection of $\vec{a}$ and $\vec{b}$ $= \dfrac{\vec{a}.\vec{b}}{|\vec{b}|}$

$= \dfrac{3}{\sqrt{6}}$

$= \dfrac{\sqrt{6}}{2}$

Q. 9. If three vectors $\vec{a}$, $\vec{b}$ and $\vec{c}$ are coplanar, prove that the vectors $\vec{a} + \vec{b}$, $\vec{b} + \vec{c}$ and $\vec{c} + \vec{a}$ are also coplanar. *[CBSE, 2014 (Comptt.)]*

Solution :

If $\vec{a}$, $\vec{b}$ and $\vec{c}$ are coplanar, then

$[\vec{a}\ \vec{b}\ \vec{c}] = 0$ ...(1)

Now, $[\vec{a} + \vec{b}, \vec{b} + \vec{c}, \vec{c} + \vec{a}]$

$= (\vec{a} + \vec{b}).\{(\vec{b} + \vec{c}) \times (\vec{c} + \vec{a})\}$

$= (\vec{a} + \vec{b}).\{\vec{b} \times \vec{c} + \vec{b} \times \vec{a} + \vec{c} \times \vec{c} + \vec{c} \times \vec{a}\}$

$= (\vec{a} + \vec{b}).\{\vec{b} \times \vec{c} + \vec{b} \times \vec{a} + \vec{c} \times \vec{a}\}$

$(\because \vec{c} \times \vec{c} = \vec{0})$

$= \vec{a}.(\vec{b} \times \vec{c}) + \vec{a}.(\vec{b} \times \vec{a}) + \vec{a}.(\vec{c} \times \vec{a}) + \vec{b}.(\vec{b} \times \vec{c}) + \vec{b}.(\vec{b} \times \vec{a}) + \vec{b}.(\vec{c} \times \vec{a})$

$= (\vec{a}\,\vec{b}\,\vec{c}) + (\vec{a}\,\vec{b}\,\vec{a}) + (\vec{a}\,\vec{c}\,\vec{a}) + (\vec{b}\,\vec{b}\,\vec{c}) + (\vec{b}\,\vec{b}\,\vec{a}) + (\vec{b}\,\vec{c}\,\vec{a})$

$= (\vec{a}\,\vec{b}\,\vec{c}) + 0 + 0 + 0 + 0 + (\vec{a}\,\vec{b}\,\vec{c})$

$= 2(\vec{a}\,\vec{b}\,\vec{c})$

$= 2 \times 0$ Using (1)

$= 0$

Hence, $\vec{a} + \vec{b}$, $\vec{b} + \vec{c}$ and $\vec{c} + \vec{a}$ are also coplanar.

Q. 10. Find the unit vector perpendicular to the plane ABC where the position vectors of A, B and C are $2\hat{i} - \hat{j} + \hat{k}$, $\hat{i} + \hat{j} + 2\hat{k}$ and $2\hat{i} + 3\hat{k}$ respectively. *[AI CBSE, 2014 (Comptt.)]*

Solution :

Let O be the origin of vectors, then

$\overrightarrow{OA} = 2\hat{i} - \hat{j} + \hat{k}$

$\overrightarrow{OB} = \hat{i} + \hat{j} + 2\hat{k}$

and $\overrightarrow{OC} = 2\hat{i} + 3\hat{k}$

$\therefore \overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA}$

$= (\hat{i} + \hat{j} + 2\hat{k}) - (2\hat{i} - \hat{j} + \hat{k})$

$= -\hat{i} + 2\hat{j} + \hat{k}$

$\overrightarrow{AC} = \overrightarrow{OC} - \overrightarrow{OA}$

$= (2\hat{i} + 3\hat{k}) - (2\hat{i} - \hat{j} + \hat{k})$

$= \hat{j} + 2\hat{k}$

$\therefore \overrightarrow{AB} \times \overrightarrow{AC} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ -1 & 2 & 1 \\ 0 & 1 & 2 \end{vmatrix}$

$= 3\hat{i} + 2\hat{j} - \hat{k}$

$\therefore$ Unit vector perpendicular to the plane ABC

$= \dfrac{\overrightarrow{AB} \times \overrightarrow{AC}}{|\overrightarrow{AB} \times \overrightarrow{AC}|} = \dfrac{3\hat{i} + 2\hat{j} - \hat{k}}{3\hat{i} + 2\hat{j} - \hat{k}}$

$= \dfrac{3\hat{i} + 2\hat{j} - \hat{k}}{\sqrt{(3)^2 + (2)^2 + (-1)^2}}$

$= \dfrac{3\hat{i} + 2\hat{j} - \hat{k}}{\sqrt{14}}$

$= \dfrac{3}{\sqrt{14}}\hat{i} + \dfrac{2}{\sqrt{14}}\hat{j} - \dfrac{1}{\sqrt{14}}\hat{k}$

Q. 11. If $\vec{a} = 3\hat{i} - \hat{j}$ and $\vec{b} = 2\hat{i} + \hat{j} - 3\hat{k}$, then express $\vec{b}$ in the from $\vec{b} = \vec{b_1} + \vec{b_2}$, where $\vec{b_1} \parallel \vec{a}$ and $\vec{b_2} \perp \vec{a}$. *[CBSE, 2013 (Comptt.)]*

Solution :

Let $\vec{b_1} = x_1\hat{i} + y_1\hat{j} + z_1\hat{k}$

and $\vec{b_2} = x_2\hat{i} + y_2\hat{j} + z_2\hat{k}$

$\therefore \vec{b} = \vec{b_1} + \vec{b_2}$

$= (x_1\hat{i} + y_1\hat{j} + z_1\hat{k}) + (x_2\hat{i} + y_2\hat{j} + z_2\hat{k})$

$= (x_1 + x_2)\hat{i} + (y_1 + y_2)\hat{j} + (z_1 + z_2)\hat{k}$ ...(1)

$\because \quad \vec{b}_1 \parallel \vec{a}$

$\therefore \quad \vec{b}_1 \times \vec{a} = \vec{0}$

$$\Rightarrow \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ x_1 & y_1 & z_1 \\ 3 & -1 & 0 \end{vmatrix} = 0\hat{i} + 0\hat{j} + 0\hat{k}$$

$\Rightarrow (z_1)\hat{i} + (3z_1)\hat{j} + (-x_1 - 3y_1)\hat{k} = 0\hat{i} + 0\hat{j} + 0\hat{k}$

$\Rightarrow \quad z_1 = 0;\ 3z_1 = 0;\ -x_1 - 3y_1 = 0$

equating the coefficients of $\hat{i}$, $\hat{j}$ and $\hat{k}$ on both sides,

$$\Rightarrow \left.\begin{matrix} z_1 = 0 \\ x_1 + 3y_1 = 0 \end{matrix}\right\} \quad \ldots(2)$$

From (1),

$2\hat{i} + \hat{j} - 3\hat{k} = (x_1 + x_2)\hat{i} + (y_1 + y_2)\hat{j} + (z_1 + z_2)\hat{k}$

$$\Rightarrow \left.\begin{matrix} x_1 + x_2 = 2 \\ y_1 + y_2 = 1 \\ z_1 + z_1 = -3 \end{matrix}\right\}$$ (Equating the coefficients of $\hat{i}$, $\hat{j}$ and $\hat{k}$ on both sides) ...(3)

Again, $\vec{b}_2 \perp \vec{a}$

$\therefore \quad \vec{b}_2 . \vec{a} = 0$

$\Rightarrow \quad (x_2\hat{i} + y_2\hat{j} + z_2\hat{k})(3\hat{i} - \hat{j}) = 0$

$\Rightarrow \quad 3x_2 - y_2 = 0 \quad \ldots(4)$

Solving (2), (3) and (4), we get

$$x_1 = \frac{3}{2}, y_1 = -\frac{1}{2}, z_1 = 0;$$

$$x_2 = \frac{1}{2}, y_2 = \frac{3}{2}, z_2 = -3.$$

$$\therefore \quad \vec{b}_1 = \frac{3}{2}\hat{i} - \frac{1}{2}\hat{j},\ \vec{b}_2 = \frac{1}{2}\hat{i} + \frac{3}{2}\hat{j} - 3\hat{k}$$

$$\therefore 2\hat{i} + \hat{j} - 2\hat{k} = \left(\frac{3}{2}\hat{i} - \frac{1}{2}\hat{j}\right) + \left(\frac{1}{2}\hat{i} + \frac{3}{2}\hat{j} - 3\hat{k}\right)$$

Q. 12. If $\vec{a}$, $\vec{b}$ and $\vec{c}$ are three vectors such that one is perpendicular to the vector obtained by sum of the other two and $|\vec{a}| = 3$, $|\vec{b}| = 4$ and $|\vec{c}| = 5$, then prove that $|\vec{a} + \vec{b} + \vec{c}| = 5\sqrt{2}$.

[*CBSE*, 2013 (*Comptt.*)]

Solution

$\because \quad \vec{a} \perp (\vec{b} + \vec{c})$

$\therefore \quad \vec{a}.(\vec{b} + \vec{c}) = 0$

$\Rightarrow \quad \vec{a}.\vec{b} + \vec{a}.\vec{c} = 0 \quad \ldots(1)$

$\because \quad \vec{b} \perp (\vec{c} + \vec{a})$

$\therefore \quad \vec{b}.(\vec{c} + \vec{a}) = 0$

$\Rightarrow \quad \vec{b}.\vec{c} + \vec{b}.\vec{a} = 0$

$\Rightarrow \quad \vec{b}.\vec{c} + \vec{a}.\vec{b} = 0 \quad \ldots(2)$

$\because \quad \vec{c} \perp (\vec{a} + \vec{b})$

$\therefore \quad \vec{c}.(\vec{a} + \vec{b}) = 0$

$\Rightarrow \quad \vec{c}.\vec{a} + \vec{c}.\vec{b} = 0$

$\Rightarrow \quad \vec{c}.\vec{a} + \vec{b}.\vec{c} = 0 \quad \ldots(3)$

Adding (1), (2) and (3), we get

$\Rightarrow \quad (\vec{a}.\vec{b} + \vec{b}.\vec{c} + \vec{c}.\vec{a}) = 0$

$\Rightarrow \quad \vec{a}.\vec{b} + \vec{b}.\vec{c} + \vec{c}.\vec{a} = 0 \quad \ldots(4)$

Now, $|\vec{a} + \vec{b} + \vec{c}|^2$

$= (\vec{a} + \vec{b} + \vec{c})^2$

$= \vec{a}^2 + \vec{b}^2 + \vec{c}^2 + 2(\vec{a}.\vec{b}) + (\vec{b}.\vec{c}) + (\vec{c}.\vec{a})$

$= 9 + 16 + 25 + 2(0) = 50$

$\Rightarrow \quad = \vec{a} + \vec{b} + \vec{c} = \sqrt{50}$

$= 5\sqrt{2}$

Q. 13. Prove that for any three vectors $\vec{a}$, $\vec{b}$ and $\vec{c}$, $(\vec{a} + \vec{b}, \vec{b} + \vec{c}, \vec{c} + \vec{a}) = (\vec{a}\ \vec{b}\ \vec{c})$.

(*CBSE*, 2014)

Solution :

$(\vec{a} + \vec{b}, \vec{b} + \vec{c}, \vec{c} + \vec{a})$

$= (\vec{a} + \vec{b}).\{(\vec{b} + \vec{c}) \times (\vec{c} + \vec{a})\}$

$= (\vec{a} + \vec{b}).\{\vec{b} \times \vec{c} + \vec{b} \times \vec{a} + \vec{c} \times \vec{c} + \vec{c} \times \vec{a}\}$

$= (\vec{a} + \vec{b}).\{\vec{b} \times \vec{c} + \vec{b} \times \vec{a} + \vec{0} + \vec{c} \times \vec{a}\}$

$= (\vec{a} + \vec{b}).\{\vec{b} \times \vec{c} + \vec{b} \times \vec{a} + \vec{c} \times \vec{a}\}$

$= \vec{a}.(\vec{b} \times \vec{c}) + \vec{a}.(\vec{b} \times \vec{a}) + \vec{a}(\vec{c} \times \vec{a}) + \vec{b}.(\vec{b} \times \vec{c}) + \vec{b}.(\vec{b} \times \vec{a}) + \vec{b}(\vec{c} \times \vec{a})$

$= (\vec{a}\ \vec{b}\ \vec{c}) + (\vec{a}\ \vec{b}\ \vec{a}) + (\vec{a}\ \vec{c}\ \vec{a}) + (\vec{a}\ \vec{a}\ \vec{c}) + (\vec{b}\ \vec{b}\ \vec{a}) + (\vec{b}\ \vec{c}\ \vec{a})$

$= (\vec{a}\ \vec{b}\ \vec{c}) + 0 + 0 + 0 + 0 + (\vec{a}\ \vec{b}\ \vec{c})$

$= 2[\vec{a}\ \vec{b}\ \vec{c}]$

Q. 14. Show that the four points A, B, C and D with position vectors $4\hat{i} + 5\hat{j} + \hat{k}$, $-\hat{j} - \hat{k}$, $3\hat{i} + 9\hat{j} + 4\hat{k}$ and $4(-\hat{i} + \hat{j} + \hat{k})$ respectively are coplanar.

(*AI CBSE*, 2014)

Solution :

Let be the origin of vectors, then

$\overrightarrow{OA} = 4\hat{i} + 5\hat{j} + \hat{k}$

$\overrightarrow{OB} = -\hat{j} - \hat{k}$

$\overrightarrow{OC} = 3\hat{i} + 9\hat{j} + 4\hat{k}$

$\overrightarrow{OD} = -4\hat{i} + 4\hat{j} + 4\hat{k}$

$\therefore \quad \overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA}$

$= (-\hat{j} - \hat{k}) - (4\hat{i} + 5\hat{j} + \hat{k})$

$= -4\hat{i} - 6\hat{j} - 2\hat{k}$

$\overrightarrow{AC} = \overrightarrow{OC} - \overrightarrow{OA}$

$= (3\hat{i} + 4\hat{j} + 4\hat{k}) - (4\hat{i} + 5\hat{j} + \hat{k})$

$= -\hat{i} + 4\hat{j} + 3\hat{k}$

and $\overrightarrow{AD} = \overrightarrow{OD} - \overrightarrow{OA}$

$= (-4\hat{i} + 4\hat{j} + 4\hat{k}) - (4\hat{i} + 5\hat{j} + \hat{k})$

$= -8\hat{i} - \hat{j} + 3\hat{k}$

Now, $(\overrightarrow{AB}, \overrightarrow{AC}, \overrightarrow{AD})$

$$= \begin{vmatrix} -4 & -6 & -2 \\ -1 & 4 & 3 \\ -8 & -1 & 3 \end{vmatrix}$$

$= -4(12+3) - 6(-24+3) - 2(1+32)$

$= -60 + 126 - 66$

$= 0$

Hence, the points A, B, C and D are coplanar.

Q. 15. If $\vec{a}$ and $\vec{b}$ are two vectors such that $|\vec{a} + \vec{b}| = |\vec{a}|$, then prove that $2\vec{a} + \vec{b}$ is perpendicular to vactor $\vec{b}$. (*CBSE*, 2013)

Solution :

$$|\vec{a} + \vec{b}| = |\vec{a}|$$

$\Rightarrow \quad |\vec{a} + \vec{b}|^2 = |\vec{a}|^2$ (squaring both sides)

$\Rightarrow \quad \vec{a}^2 + \vec{b}^2 + 2\vec{a}.\vec{b} = \vec{a}$

$\Rightarrow \quad \vec{b}^2 + 2\vec{a}.\vec{b} = 0$

$\Rightarrow \quad \vec{b}.\vec{b} + \vec{b}.(2\vec{a}) = 0$

$\Rightarrow \quad \vec{b}.(\vec{b} + 2\vec{a}) = 0$

$\Rightarrow \quad \vec{b}.(2\vec{a} + \vec{b}) = 0$

$\Rightarrow \quad \vec{b} \perp (2\vec{a} + \vec{b})$

$\Rightarrow \quad (2\vec{a} + \vec{b}) \perp \vec{b}$

Q. 16. Vectors $\vec{a}, \vec{b}$ and $\vec{c}$ are such that $\vec{a} + \vec{b} + \vec{c} = \vec{0}$ and $|\vec{a}| = 3$, $|\vec{b}| = 5$ and $|\vec{c}| = 7$. Find the angle between $\vec{a}$ and $\vec{b}$. (*CBSE*, 2014)

Solution :

$$\vec{a} + \vec{b} + \vec{c} = \vec{0}$$

$\Rightarrow \quad \vec{a} + \vec{b} = -\vec{c}$

$\Rightarrow \quad (\vec{a} + \vec{b}) = (-\vec{c})^2$

$\Rightarrow \vec{a}.\vec{a} + 2\vec{a}.\vec{b} + \vec{b}.\vec{b} = \vec{c}^2$

$\Rightarrow \quad a^2 + 2\vec{a}.\vec{b} + b^2 = \vec{c}^2$

$\Rightarrow \quad a + 2\vec{a}.\vec{b} + 25 = 49$

$\Rightarrow \quad 34 + 2\vec{a}.\vec{b} = 49$

$\Rightarrow \quad 34 + 2|\vec{a}||\vec{b}|\cos\theta = 49$

$\Rightarrow \quad 34 + 2.3.5\cos\theta = 49$

$\Rightarrow \quad 34 + 30\cos\theta = 49$

$\Rightarrow \quad 30\cos\theta = 15$

$\Rightarrow \quad \cos\theta = \frac{15}{30} = \frac{1}{2} = \cos\frac{\pi}{3}$

$\Rightarrow \quad \theta = \frac{\pi}{3}$

Hence, the angle between $\vec{a}$ and $\vec{b}$ is $\frac{\pi}{3}$.

Q. 17. If $\vec{a} = 2\hat{i} - \hat{j} + \hat{k}$ and $\vec{b} = \hat{i} + \hat{j} + \hat{k}$, then find the following : (*JAC*, 2015)

(i) $|\vec{a}|$ **(ii) $\vec{a}.\vec{b}$**

(iii) $\vec{a} \times \vec{b}$ **(iv) Projection of $\vec{b}$ on $\vec{a}$**

Solution :

$\because \therefore \quad |\vec{a}| = 2\hat{i} - \hat{j} + \hat{k}$ and $\vec{b} = \hat{i} + \hat{j} + \hat{k}$

(i) $|\vec{a}| = |2\hat{i} - \hat{j} + \hat{k}|$

$= \sqrt{(2)^2 + (-1)^2 + (1)^2}$

$= \sqrt{4+1+1}$

$= \sqrt{6}$

(ii) $\vec{a}.\vec{b} = (2\hat{i} - \hat{j} + \hat{k}).(\hat{i} + \hat{j} + \hat{k})$

$= 2 - 1 + 1$

$= 2$

(iii) $$\vec{a} \times \vec{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & -1 & 1 \\ 1 & 1 & 1 \end{vmatrix}$$

$= \hat{i}(-1-1) - \hat{j}(2-1) + \hat{k}(2+1)$

$= -2\hat{i} - \hat{j} + 3\hat{k}$

$\therefore \quad |\vec{a} \times \vec{b}| = |-2\hat{i} - \hat{j} + 3\hat{k}|$

$= \sqrt{(-2)^2 + (-1)^2 + (3)^2}$

$= \sqrt{4+1+9}$

$= \sqrt{14}$

(iv) Projection of $\vec{b}$ on $\vec{a}$ $= \vec{b}.\frac{\vec{a}}{|\vec{a}|}$

$= (\hat{i} + \hat{j} + \hat{k}) \cdot \frac{(2\hat{i} - \hat{j} + \hat{k})}{|2\hat{i} - \hat{j} + \hat{k}|}$

$$= \frac{2-1+1}{\sqrt{6}}$$

$$= \frac{2}{\sqrt{6}}$$

NCERT QUESTIONS

Q. 1. Find a vector of magnitude 5 units and parallel to the resultant of the vectors $\vec{a} = 2\hat{i}+3\hat{j}-\hat{k}$ and $\vec{b} = \hat{i}-2\hat{j}+\hat{k}$. (*CBSE*, 2011)

Solution :

Let $\bar{c}$ is a vector such that $\vec{c} = c_1\hat{i} + c_2\hat{j} + c_3\hat{k}$

and $|\bar{c}| = 5 \Rightarrow \sqrt{c_1^2+c_2^2+c_3^2} = 5$

$\Rightarrow \quad c_1^2+c_2^2+c_3^2 = 25$ ...(1)

Thus, the resultant of $\bar{a}$ and $\bar{b} = 3\hat{i}+\hat{j}+0\hat{k} = \bar{d}$ (Let)

Now $\bar{c}$ and $\bar{d}$ are parallel to each other, then

$$\bar{c}\times\bar{d} = \vec{0}$$

$$\Rightarrow \begin{vmatrix} i & j & k \\ c_1 & c_2 & c_3 \\ 3 & 1 & 0 \end{vmatrix} = \vec{0}$$

$\Rightarrow \hat{i}\,(0-c_3) + \hat{j}\,(3c_3-0) + \hat{k}\,(c_1-3c_2) = \vec{0}$

$\Rightarrow c_3 = 0$ and $c_1 = 3c_2$

From (1),

$$(3c_2)^2 + c_2^2 + 0 = 25$$

$$\Rightarrow \quad 9c_2^2 + c_2^2 = 25$$

$$\Rightarrow \quad c_2^2 = \frac{25}{10} \Rightarrow c_2 = \pm\sqrt{\frac{5}{2}}$$

$$\Rightarrow \quad c_1 = 3c_2 = \pm 3\sqrt{\frac{5}{2}}$$

Thus, required vector is $\pm\sqrt{\frac{5}{2}}(3\hat{i}+\hat{j})$.

Q. 2. Let $a = \hat{i}+4\hat{j}+2\hat{k}, \bar{b} = 3\hat{i}-2\hat{j}+7\hat{k}$ and $\vec{c} = 2\hat{i}-4\hat{j}+4\hat{k}$. Find a vector d which is $\perp$ to both $\bar{a}$ and $\bar{b}$, and $\bar{c}.\bar{d} = 15$.

Solution :

Let $\quad \bar{d} = d_1\hat{i}+d_2\hat{j}+d_3\hat{k}$

$$\hat{d}\perp\bar{a} \Rightarrow \bar{d}.\bar{a} = 0$$

$\Rightarrow \quad d_1 + 4d_2 + 2d_3 = 0$ ...(1)

$\Rightarrow \quad 3d_1 - 2d_2 + 7d_3 = 0$ ...(2)

$\quad \bar{c}.\bar{d} = 15$ (Given)

$\Rightarrow \quad 2d_1 - d_2 + 4d_3 = 15$ ...(3)

Solving equations (1), (2) and (3), we get

$$d_1 = \frac{160}{3},\ d_2 = -\frac{5}{3} \text{ and } d_3 = -\frac{70}{3}$$

Hence, required vector $\bar{d} = \frac{1}{3}[160\hat{i} - 5\hat{i} + 70\hat{k}]$.

Q. 3. Find a unit vector perpendicular to each of the vectors $\bar{a}+\bar{b}$ and $\bar{a}-\bar{b}$, where $3\hat{i}+2\hat{j}+2\hat{k}$ and $\bar{b} = \hat{i}+2\hat{j}-2\hat{k}$. (*CBSE*, 2011)

Solution :

$$\bar{a}+\bar{b} = 4\hat{i}+4\hat{j}+0\hat{k} = \bar{c} \quad \text{(Let)}$$

$$\bar{a}-\bar{b} = 2\hat{i}+0\hat{j}+4\hat{k} = \bar{d}$$

then unit vector which is perpendicular to $\bar{c}$ and $\bar{d}$,

$$= \frac{(\bar{c}\times\bar{d})}{|\bar{c}\times\bar{d}|}$$

$$(\bar{c}\times\bar{d}) = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 4 & 4 & 0 \\ 2 & 0 & 4 \end{vmatrix}$$

$$= \hat{i}(16-0) - \hat{j}(16-0) + \hat{k}\,(0-8)$$

$$= 16\hat{i}-16\hat{j}-8\hat{k}$$

and $\quad |\bar{c}\times\bar{d}| = \sqrt{256+256+64} = \sqrt{576} = 24$

Thus, required vector $= \dfrac{16\hat{i}-16\hat{j}-8\hat{k}}{24}$

$$= \left(\frac{2\hat{i}-2\hat{j}-\hat{k}}{3}\right).$$

Q. 4. Find the scalar components of a unit vector which is perpendicular to each of the vectors $(\hat{i}+2\hat{j}-\hat{k})$ and $(3\hat{i}-\hat{j}+2\hat{k})$.

Solution

$\vec{a} = a_1\hat{i}+a_2\hat{j}+a_3\hat{k}$ be a unit vector, which is perpendicular to each of the vectors $(\hat{i}+2\hat{j}-\hat{k})$ and $(3\hat{i}-\hat{j}+2\hat{k})$.

$\because \vec{a}$ is an unit vector.

$\therefore \quad a_1^2+a_2^2+a_3^2 = 1$

$\because \vec{a}$ is perpendicular to each of the vectors $\hat{i}+2\hat{j}-\hat{k}$ and $3\hat{i}-\hat{j}+2\hat{k}$, then

$$a_1 + 2a_2 - a_3 = 0 \quad ...(2)$$

$$3a_1 - a_2 + 2a_3 = 0 \quad ...(3)$$

Solving eqns. (2) and (3),

$$a_2 = -\frac{5}{3}a_1 \text{ and } a_3 = -\frac{7}{3}a_1$$

putting the value of a_2 and a_3 in eq. (1),

$$a_1^2 + \frac{25}{9}a_1^2 + \frac{49}{9}a_1^2 = 1$$

$$\Rightarrow \quad \frac{83}{9}a_1^2 = 1 \text{ or } a_1 = \pm\frac{3}{\sqrt{83}}$$

$$\Rightarrow \quad a_2 = \pm\frac{5}{\sqrt{83}} \text{ and } a_3 = \pm\frac{7}{\sqrt{83}}$$

Sector components are $\frac{3}{\sqrt{83}}, -\frac{5}{\sqrt{83}}, -\frac{7}{\sqrt{83}}$ or $-\frac{3}{\sqrt{83}}, \frac{5}{\sqrt{83}}, \frac{7}{\sqrt{83}}$.

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