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DEFINITE INTEGRALS

IMPORTANT FORMULAE

- **Fundamental theorem of calculus :**

$$\int_a^b f(x)dx = F(b) - F(a), \text{where } \frac{d}{dx}F(x) = f(x)$$

Properties of Definite Integration

- $\int_a^b f(x)dx = \int_a^b f(y)dy$
- $\int_a^b f(x)dx = -\int_b^a f(x)dx$
- $\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx, a < c < b$
- $\int_0^a f(x)dx = \int_0^a f(a-x)dx$
- $\int_a^b f(x)dx = \int_a^b f(a+b-x)dx$
- $\int_{-a}^a f(x)dx = \begin{cases} 0, & \text{when } f(-x) = -f(x) \\ \int_{-a}^a f(x)d(x), & \text{when } f(2a-x) = f(x) \end{cases}$
- $\int_0^{2a} f(x)dx = \begin{cases} 0, & \text{when } f(2a-x) = -f(x) \\ 2\int_0^a f(x)d(x), & \text{when } f(2a-x) = f(x) \end{cases}$

Multiple Choice Questions

1. $\int_0^{\pi/2} \frac{\sqrt{\cos x}}{\sqrt{\sin x}+\sqrt{\cos x}}dx =$ (*BSEB*, 2013)

(a) $\frac{\pi}{4}$ (b) $-\frac{\pi}{4}$ (c) 0 (d) $\frac{\pi}{2}$

2. $\int_0^{\pi/4} \log(1+\tan x)\,dx =$ (*BSEB*, 2013)

(a) $\frac{\pi}{8}\log 2$ (b) $\frac{\pi}{4}\log 2$ (c) $\frac{\pi}{2}\log 2$ (d) 0

3. $\int_0^{\frac{\pi}{2}} \sin 2x \log(\tan x)dx =$ (*BSEB*, 2013)

(a) 0 (b) $\frac{\pi}{2}$ (c) $\frac{\pi}{4}$ (d) $-\frac{\pi}{2}$

4. $\int_0^1 \frac{x}{x^2+1}dx =$

(a) $-\frac{1}{2}\log 2$ (b) $\log\frac{1}{2}$ (c) $\log 2$ (d) $\frac{1}{2}\log 2$

5. $\int_0^{\pi/4} \sec^2 x\,dx =$

(a) 0 (b) -1 (c) 1 (d) 2

6. $\int_0^1 \frac{(\tan^{-1}x)^2}{1+x^2}dx =$ (*BSEB*, 2013)

(a) 1 (b) $\frac{\pi^3}{64}$

(c) $\frac{\pi^3}{192}$ (d) none of these

7. $\int_{-2}^2 |x|\,dx =$ (*BSEB*, 2010)

(a) 0 (b) 2 (c) 1 (d) 4

8. $\int_{-\pi/2}^{\pi/2}(x^3 + x\cos x + \tan^5 x + 1)dx =$

(a) $\frac{\pi}{2}$ (b) π (c) 0 (d) 2

9. $\int_0^{\pi/2} \log\sin x\,dx =$

(a) $\frac{\pi}{2}\log\frac{1}{2}$ (b) $\frac{\pi}{2}\log 2$

(c) $\pi\log 2$ (d) none of these

10. $\int_2^3 \frac{1}{x\log x}dx =$

(a) log 6 (b) log 3

(c) log 2 (d) log log 3 – log log 2

11. $\int_a^b x^5 dx =$ (*BSEB*, 2015)

(a) $b^5 - a^5$ (b) $\frac{b^6-a^6}{6}$ (c) $\frac{a^6-b^6}{6}$ (d) $a^5 - b^5$

Ans. 1. (a), 2. (a), 3. (a), 4. (d), 5. (c), 6. (d), 7. (a), 8. (b), 9. (a), 10. (d), 11. (b).

Very Short Answer Type Questions

Q. 1. If $f(x) = \int_0^x t\sin t\,dt$, then write the value of $f'(x)$. (*AI CBSE*, 2014)

Solution : $f(x) = \int_0^x t\sin t\,dt = [t\cos t]_0^x - \int_0^x \cos t\,dt$

$\Rightarrow \quad = x\cos x + [\sin t]_0^x$

$\Rightarrow \quad = x\cos x + \sin x$

Q. 2. Evaluate : $\int_1^2 \frac{x^3-1}{x^2}dx$

[*AI CBSE*, 2014 (*Comptt.*)]

Solution : $I = \int_1^2 \frac{x^3-1}{x^2}dx$

$= \int_1^2\left(x - \frac{1}{x^2}\right)dx$

$= \left(\frac{x^2}{2} + \frac{1}{x}\right)_1^2$

$= \left(2+\frac{1}{2}\right) - \left(\frac{1}{2}+1\right)$

$= 1$

Q. 3. Evaluate : $\int_e^{e^2} \frac{dx}{x\log x}$ (*AI CBSE*, 2014)

Solution : $I = \int_e^{e^2} \frac{dx}{x \log x}$

Put $\log x = t \Rightarrow \frac{1}{x}dx = dt$

$= \int_1^2 \frac{dt}{t}$

$= (\log t)_1^2$

$= \log 2 - \log 1$

$= \log 2 - 0$

$= \log 2$

Q. 4. If $\int_0^a \frac{1}{4+x^2}\,dx = \frac{\pi}{8}$, find the value of a. (*AI CBSE*, 2014)

Solution :

We have $\int_0^a \frac{1}{4+x^2}dx = \frac{\pi}{8}$

$\Rightarrow \int_0^a \frac{1}{x^2+2^2}dx = \frac{\pi}{8}$

$\Rightarrow \frac{1}{2}\left[\tan^{-1}\left(\frac{x}{2}\right)\right]_0^a = \frac{\pi}{8}$

$\Rightarrow \frac{1}{2}\left[\tan^{-1}\frac{a}{2} - \tan^{-1}0\right] = \frac{\pi}{8}$

$\Rightarrow \frac{1}{2}\left(\tan^{-1}\frac{a}{2} - 0\right) = \frac{\pi}{8}$

$\Rightarrow \tan^{-1}\frac{a}{2} = \frac{\pi}{4}$

$\Rightarrow \tan^{-1}\frac{a}{2} = \tan^{-1}1$

$\Rightarrow \frac{a}{2} = 1$

$\therefore a = 2$

Q. 5. Evaluate : $\int_0^3 \frac{dx}{9+x^2}$ (*CBSE*, 2014)

Solution : $I = \int_0^3 \frac{dx}{9+x^2}$

$= \int_0^3 \frac{dx}{x^2+3^2}$

$= \frac{1}{3}\left[\tan^{-1}\left(\frac{x}{3}\right)\right]_0^3$

$= \frac{1}{3}[\tan^{-1}(1) - \tan^{-1}(0)]$

$= \frac{1}{3}\left(\frac{\pi}{4} - 0\right)$

$= \frac{\pi}{12}$

Q. 6. Evaluate : $\int_0^1 \frac{\tan^{-1}x}{1+x^2}dx$ [*AI CBSE*, 2014 (*Comptt.*)]

Solution : $I = \int_0^1 \frac{\tan^{-1}x}{1+x^2}dx$

Put $\tan^{-1}x = 1 \Rightarrow \frac{1}{1+x^2}dx = dt$

when $x = 0, t = \tan^{-1}0 = 0$

and when $x = 1, t = \tan^{-1}1 = \frac{\pi}{4}$

$\Rightarrow I = \int_0^{\pi/4} t\,dt$

$= \left(\frac{t^2}{2}\right)_0^{\pi/4} = \frac{1}{2}\left[\left(\frac{\pi}{4}\right)^2 - (0)^2\right]$

$= \frac{\pi^2}{32}$

Q. 7. Evaluate : $\int_5^{17} \frac{x}{x^2+1}dx$ (*AI CBSE*, 2014)

Solution : $I = \int_0^1 \frac{x}{x^2+1}dx$

Put $x^2 + 1 = t$

$\Rightarrow 2x\,dx = dt$

$\Rightarrow x\,dx = \frac{1}{2}dt$

$\Rightarrow I = \frac{1}{2}\int_5^{17}\frac{dt}{t}$

$= \frac{1}{2}[\log t]_5^{17}$

$= \frac{1}{2}(\log 17 - \log 5)$

$= \frac{1}{2}\log\left(\frac{17}{5}\right)$

Q. 8. Evaluate : $\int_{-\pi/2}^{\pi/2} \sin^2 x\,dx$ (*BSEB*, 2013)

Solution : $I = \int_{-\pi/2}^{\pi/2} \sin^2 x\,dx$

$= 2\int_0^{\pi/2} \sin^2 x\,dx$

$[\because \sin^2(-x) = \sin^2 x]$

$= \int_0^{\pi/2} (1 - \cos 2x)\,dx$

$= \left(x - \frac{\sin 2x}{2}\right)_0^{\pi/2}$

$= \left(\frac{\pi}{2} - \frac{\sin\pi}{2}\right) - \left(0 - \frac{\sin 0}{2}\right)$

$= \frac{\pi}{2}$

Short Answer Type Questions

Q. 1. Evaluate : $\int_0^{\pi/2} \frac{\sin x}{\sin x + \cos x}dx$ (*JAC*, 2013)

Solution : $I = \int_0^{\pi/2} \frac{\sin x}{\sin x + \cos x}dx$ ...(1)

$$\Rightarrow \quad I = \int_0^{\pi/2} \frac{\sin\left(\frac{\pi}{2}-x\right)}{\sin\left(\frac{\pi}{2}-x\right)+\cos\left(\frac{\pi}{2}-x\right)}dx$$

$$\left(\because \int_0^a f(x)dx = \int_0^a f(a-x)dx\right)$$

$$\Rightarrow \quad = \int_0^{\pi/2} \frac{\cos x}{\cos x+\sin x}dx$$

$$\Rightarrow \quad = \int_0^{\pi/2} \frac{\cos x}{\sin x+\cos x}dx \quad \ldots(2)$$

Adding (1) and (2), we get

$$2I = \int_0^{\pi/2} \frac{\sin x+\cos x}{\sin x+\cos x}dx$$

$$\Rightarrow \quad = \int_0^{\pi/2} 1dx$$

$$= (x)_0^{\pi/2} = \frac{\pi}{2} - 0 = \frac{\pi}{2}$$

$$\Rightarrow \quad I = \frac{\pi}{4}$$

Q. 2. Evaluate : $\int_0^{\pi/2} e^x(\sin x - \cos x)dx$

(CBSE, 2014)

Solution : $I = \int_0^{\pi/2} e^x(\sin x - \cos x)dx$

$$= \int_0^{\pi/2} e^x \sin x\, dx - \int_0^{\pi/2} e^x \cos x\, dx$$

$$= \left[e^x(-\cos x)\right]_0^{\pi/2} - \int_0^{\pi/2} e^x(-\cos x)dx - \int_0^{\pi/2} e^x \cos x\, dx$$

$$= -\left[e^x \cos x\right]_0^{\pi/2}$$

$$= -\left[e^{\pi/2}\cos\frac{\pi}{2}\right] + e^0 \cos 0$$

$$= -\left(e^{\pi/2}.0\right) + 1(1)$$

$$= 1$$

Q. 3. Evaluate : $\int_1^2 \frac{xe^x}{(1+x)^2}dx$

(BSER, 2014)

Solution : $= \int_1^2 \frac{xe^x}{(1+x)^2}dx$

$$= \int \frac{(1+x-1)e^x}{(1+x)^2}dx$$

$$= \int \frac{e^x}{1+x}dx - \int \frac{e^x}{(1+x)^2}dx$$

$$= \frac{e^x}{1+x} - \int -\frac{1}{(1+x)^2}e^x\, dx - \int \frac{e^x}{(1+x)^2}dx + C$$

$$= \frac{e^x}{1+x} + C$$

$$\therefore \quad \int_1^2 \frac{xe^x}{(1+x)^2}dx = \left(\frac{e^x}{1+x}+C\right)_1^2$$

$$= \frac{e^2}{1+2} - \frac{e^1}{1+1}$$

$$= \frac{e^2}{3} - \frac{e}{2}$$

Q. 4. Evaluate : $\int_{\pi/2}^{\pi} \frac{1-\sin x}{1-\cos x}dx$ *(JAC, 2013)*

Solution : $I = \int_{\pi/2}^{\pi} \frac{1-\sin x}{1-\cos x}dx$

$$= \int_{\pi/2}^{\pi} \frac{1}{1-\cos x}dx - \int_{\pi/2}^{\pi} \frac{\sin x}{1-\cos x}dx$$

$$= \int_{\pi/2}^{\pi} \frac{1}{2\sin^2\frac{x}{2}}dx - \int_{\pi/2}^{\pi} \frac{2\sin\frac{x}{2}\cos\frac{x}{2}}{2\sin^2\frac{x}{2}}dx$$

$$= \frac{1}{2}\int_{\pi/2}^{\pi} \operatorname{cosec}^2\frac{x}{2}dx - \int_{\pi/2}^{\pi} \cot\frac{x}{2}dx$$

$$= \left(-\cot\frac{x}{2}\right)_{\pi/2}^{\pi} - 2\left(\log \sin\frac{x}{2}\right)_{\pi/2}^{\pi}$$

$$= -\cot\frac{\pi}{2} + \cot\frac{\pi}{4} - 2\left[\log \sin\frac{\pi}{2} - \log \sin\frac{\pi}{4}\right]$$

$$= 0 + 1 - 2\left[\log 1 - \log\frac{1}{\sqrt{2}}\right]$$

$$= 1 + 2\log\frac{1}{\sqrt{2}}$$

$$= 1 + 2(\log 1 - \log\sqrt{2})$$

$$= 1 + 2\left(0 - \frac{1}{2}\log 2\right)$$

$$= 1 - \log 2$$

Q. 5. Evaluate : $\int_0^{2\pi} \frac{1}{1+e^{\sin x}}dx$ *(AI CBSE, 2013)*

Solution : $I = \int_0^{2\pi} \frac{1}{1+e^{\sin x}}dx \quad (1)$

$$\Rightarrow \quad I = \int_0^{2\pi} \frac{1}{1+e^{\sin(2\pi-x)}}dx$$

$$\left(\because \int_0^a f(x)dx = \int_0^a f(a-x)dx\right)$$

$$\Rightarrow \quad I = \int_0^{2\pi} \frac{1}{1+e^{-\sin x}}dx$$

$$\Rightarrow \quad I = \int_0^{2\pi} \frac{e^{\sin x}}{1+e^{\sin x}}dx \quad \ldots(2)$$

Adding (1) and (2), we get

$$2I = \int_0^{2\pi} \frac{1+e^{\sin x}}{1+e^{\sin x}}dx$$

$$= \int_0^{2\pi} 1\,dx$$

$$= (x)_0^{2\pi} = 2\pi$$

$$\Rightarrow \quad I = \pi$$

Q. 6. Evaluate : $\int_0^{\pi/2} \frac{\cos x}{(1+\sin x)(2+\sin x)}dx$

(BSER, 2013)

Solution : $I = \int_0^{\pi/2} \frac{\cos x}{(1+\sin x)(2+\sin x)}dx$

Put $\sin x = t$
$\Rightarrow \cos x\, dx = dt$

$$= \int_0^1 \frac{dt}{(1+t)(2+t)}$$

$$= \int_0^1 \left(\frac{1}{1+t} - \frac{1}{2+t}\right) dt$$

(Resolving the integrand into path of freedom)

$$= [\log(1+t) - \log(2+t)]_0^1$$

$$= \left[\log\left(\frac{1+t}{2+t}\right)\right]_0^1$$

$$= \log\frac{2}{3} - \log\frac{1}{2}$$

$$= \log\left(\frac{\frac{2}{3}}{\frac{1}{2}}\right) = \log\frac{4}{3}$$

Q. 7. Evaluate : $\int_0^{\pi/4} (\tan x - x)\tan^2 x\, dx$

(BSEB, 2013)

Solution : $I = \int_0^{\pi/4} (\tan x - x)\tan^2 x\, dx$

$$= \int_0^{\pi/4} (\tan x - x)(\sec^2 x - 1)\, dx$$

$$= \int_0^{\pi/4} (\tan x - x)\sec^2 x\, dx - \int_0^{\pi/4} (\tan x - x)\, dx$$

$$= \int_0^{\pi/4} \tan x \sec^2 x\, dx - \int_0^{\pi/4} x \sec^2 dx$$

$$- \left(\log \sec x - \frac{x^2}{2}\right)_0^{\pi/4} \quad \ldots(1)$$

$$I_1 = \int_0^{\pi/4} \tan x \sec^2 x\, dx$$

Put $\tan x = t \Rightarrow \sec^2 x\, dx = dt$

$$= \int_0^1 t\, dt = \left(\frac{t^2}{2}\right)_0^1 = \frac{1}{2} \quad \ldots(2)$$

$$I_2 = \int_0^{\pi/4} \underset{I}{x} \underset{II}{\sec^2 x}\, dx$$

$$= (x\tan x)_0^{\pi/4} - \int_0^{\pi/4} 1.\tan x\, dx$$

$$= \frac{\pi}{4}\tan\frac{\pi}{4} - 0 - (\log\sec x)_0^{\pi/4}$$

$$= \frac{\pi}{4}\tan\frac{\pi}{4} - (\log\sec\frac{\pi}{4} - \log\sec 0)$$

$$= \frac{\pi}{4} - (\log\sqrt{2} - 0)$$

$$= \frac{\pi}{4} - \frac{1}{2}\log 2 \quad \ldots(3)$$

$$\left(\log\sec x - \frac{x^2}{2}\right)_0^{\pi/4} = \left\{\log\sec\frac{\pi}{4} - \frac{1}{2}\left(\frac{\pi}{4}\right)^2\right\} - \{\log\sec 0 - 0\}$$

$$= \log\sqrt{2} - \frac{\pi^2}{32}$$

$$= \frac{1}{2}\log 2 - \frac{\pi^2}{32} \quad \ldots(4)$$

Using equations (2), (3) and (4), we get from (1)

$$I = \frac{1}{2} - \left(\frac{\pi}{4} - \frac{1}{2}\log 2\right) - \left(\frac{1}{2}\log 2 - \frac{\pi^2}{32}\right)$$

$$= \frac{1}{2} - \frac{\pi}{4} + \frac{1}{2}\log 2 - \frac{1}{2}\log 2 - \frac{\pi^2}{32}$$

$$= \frac{1}{2} - \frac{\pi}{4} + \frac{\pi^2}{32}$$

Q. 8. Evaluate : $\int_{\pi/6}^{\pi/3} \frac{dx}{1+\sqrt{\cot x}}$

(CBSE, 2014)

Solution : $I = \int_{\pi/6}^{\pi/3} \frac{dx}{1+\sqrt{\cot x}}$

$$\Rightarrow I = \int_{\pi/6}^{\pi/3} \frac{\sqrt{\sin x}}{\sqrt{\sin x}+\sqrt{\cos x}} dx \quad \ldots(1)$$

$$\Rightarrow I = \int_{\pi/6}^{\pi/3} \frac{\sqrt{\sin\left(\frac{\pi}{6}+\frac{\pi}{3}-x\right)}}{\sqrt{\sin\left(\frac{\pi}{6}+\frac{\pi}{3}-x\right)}+\sqrt{\cos\left(\frac{\pi}{6}+\frac{\pi}{3}-x\right)}} dx$$

$$\left(\because \int_a^b f(x)dx = \int_a^b f(a+b-x)dx\right)$$

$$\Rightarrow I = \int_{\pi/6}^{\pi/3} \frac{\sqrt{\cos x}}{\sqrt{\cos x}+\sqrt{\sin x}} dx$$

$$\Rightarrow I = \int_{\pi/6}^{\pi/3} \frac{\sqrt{\cos x}}{\sqrt{\sin x}+\sqrt{\cos x}} dx \quad \ldots(2)$$

Adding (1) and (2), we get

$$2I = \int_{\pi/6}^{\pi/3} \frac{\sqrt{\sin x}+\sqrt{\cos x}}{\sqrt{\sin x}+\sqrt{\cos x}} dx$$

$$= \int_{\pi/6}^{\pi/3} 1\, dx$$

$$= (x)_{\pi/6}^{\pi/3}$$

$$= \frac{\pi}{3} - \frac{\pi}{6}$$

$$= \frac{\pi}{6}$$

$$\Rightarrow I = \frac{\pi}{12}$$

Q. 9. Using properties of definite integrals, evaluate the following :

$$\int_0^{\pi} \frac{4x\sin x}{1+\cos^2 x} dx$$

(AI CBSE, 2014)

Solution :

Let $I = \int_0^{\pi} \frac{4x\sin x}{1+\cos^2 x} dx$

$$\Rightarrow I = \int_0^{\pi} \frac{4(\pi-x)\sin(\pi-x)}{1+\cos^2(\pi-x)} dx$$

$$\Rightarrow I = \int_0^{\pi} \frac{4(\pi-x)\sin x}{1+\cos^2 x} dx$$

$$\Rightarrow I = 4\pi\int_0^{\pi} \frac{\sin x}{1+\cos^2 x} dx - \int_0^{\pi} \frac{4x\sin x}{1+\cos^2 x} dx$$

$$\Rightarrow I = 4\pi\int_0^{\pi} \frac{\sin x}{1+\cos^2 x} dx - I$$

$$\Rightarrow \quad 2I = 4\pi\int_0^{\pi} \frac{\sin x}{1+\cos^2 x}dx$$

Put $\cos x = t \Rightarrow -\sin x\, dx = dt$

when $x = 0 \Rightarrow t = 1$

and $x = \pi \Rightarrow t = -1$

$$\therefore \quad 2I = 4\pi\int_1^{-1} \frac{-dt}{1+t^2}$$

$$\Rightarrow \quad I = 2\pi\,[\tan^{-1} t]_1^{-1}$$

$$\Rightarrow \quad I = -\frac{\pi}{2}[\tan^{-1}(-1) - \tan^{-1}(1)]$$

$$\Rightarrow \quad I = -2\pi\left[-\frac{\pi}{4} - \frac{\pi}{4}\right]$$

$$\therefore \quad I = \pi^2$$

Q. 10. Evaluate : $\displaystyle\int_{\pi/6}^{\pi/3} \frac{dx}{1+\sqrt{\tan x}}$

(*AI CBSE*, 2011; *BSEB*, 2014)

Solution :

$$I = \int_{\pi/6}^{\pi/3} \frac{dx}{1+\sqrt{\tan x}}$$

$$= \int_{\pi/6}^{\pi/3} \frac{\sqrt{\cos x}}{1\sqrt{\sin x}+\sqrt{\cos x}}dx \quad ...(1)$$

$$\Rightarrow \quad I = \int_{\pi/6}^{\pi/3} \frac{\sqrt{\cos\left(\frac{\pi}{6}+\frac{\pi}{3}-x\right)}}{\sqrt{\sin\left(\frac{\pi}{6}+\frac{\pi}{3}-x\right)}+\sqrt{\cos\left(\frac{\pi}{6}+\frac{\pi}{3}-x\right)}}dx$$

$$= \int_{\pi/6}^{\pi/3} \frac{\sqrt{\sin x}}{\sqrt{\cos x}+\sqrt{\sin x}}$$

$$= \int_{\pi/6}^{\pi/3} \frac{\sqrt{\sin x}}{\sqrt{\sin x}+\sqrt{\cos x}}dx \quad ...(2)$$

Adding (1) and (2), we get

$$2I = \int_{\pi/6}^{\pi/3} \frac{\sqrt{\cos x}+\sqrt{\sin x}}{\sqrt{\sin x}+\sqrt{\cos x}}dx$$

$$= \int_{\pi/6}^{\pi/3} 1\,dx$$

$$= \int_{\pi/6}^{\pi/3} 1\,dx = (x)_{\pi/6}^{\pi/3} = \frac{\pi}{3} - \frac{\pi}{6}$$

$$\Rightarrow \quad I = \frac{\pi}{12}$$

Q. 11. Evaluate : $\displaystyle\int_{\pi/6}^{\pi/3} \frac{\sin x + \cos x}{\sqrt{\sin^2 x}}dx$

[*AI CBSE*, 2014 (*Comptt.*)]

Solution :

$$I = \int_{\pi/6}^{\pi/3} \frac{\sin x + \cos x}{\sqrt{\sin^2 x}}dx$$

Put $\sin x - \cos x = t \therefore (\cos x + \sin x)\, dx = dt$

when $x = \frac{\pi}{6}$, $t = \sin\frac{\pi}{6} - \cos\frac{\pi}{6} = \frac{1}{2} - \frac{\sqrt{3}}{1} = 1\frac{\sqrt{3}}{2}$;

$$= (\sin^{-1} t)_{\frac{1-\sqrt{3}}{2}}^{\frac{\sqrt{3}-1}{2}}$$

when $x = \frac{\pi}{3}$, $t = \sin\frac{\pi}{3} - \cos\frac{\pi}{3} = \frac{\sqrt{3}}{2} - \frac{1}{2} = \frac{\sqrt{3}-1}{2}$;

$$= \sin\left(\frac{\sqrt{3}-1}{2}\right) - \sin^{-1}\left(\frac{1-\sqrt{3}}{2}\right)$$

squaring $\sin x - \cos x = t$, we get

$$= \sin\left(\frac{\sqrt{3}-1}{2}\right) + \sin^{-1}\left(\frac{\sqrt{3}-1}{2}\right)$$

$\sin^2 x + \cos^2 x - 2\sin x\cos x = t^2 \Rightarrow 1 - \sin 2x = t^2$

$$= 2\sin^{-1}\left(\frac{\sqrt{3}-1}{2}\right) \qquad \Rightarrow 1 - \sin 2x = 1 - t^2$$

Q. 12. Evaluate :

$\displaystyle\int_2^5 [|x-2| + |x-3| + |x-5|]\,dx$ (*CBSE*, 2013)

Solution :

$$I = \int_2^5 [|x-2| + |x-3| + |x-5|]\,dx$$

$$= \int_2^5 |x-2|\,dx + \int_2^5 |x-3|\,dx + \int_2^5 |x-5|\,dx$$

$$= \int_2^5 |x-2|\,dx + \int_2^3 |x-3|\,dx + \int_2^5 |x-3|\,dx + \int_2^5 |x-5|\,dx$$

$$= \int_2^5 (x-2)dx + \int_2^3 -(x-3)dx + \int_3^5 (x-3)dx + \int_2^5 -(x-5)dx$$

$$= \left(\frac{x^2}{2} - 2x\right)_2^5 - \left(\frac{x^2}{2} - 3x\right)_2^3 + \left(\frac{x^2}{2} - 3x\right)_2^5 - \left(\frac{x^2}{2} - 5x\right)_2^5$$

$$= \left[\left(\frac{25}{2} - 10\right) - (2-4)\right] - \left[\left(\frac{9}{2} - 9\right) - (2-6)\right] + \left[\left(\frac{25}{2} - 15\right) - \left(\frac{9}{2} - 9\right)\right] - \left[\left(\frac{29}{2} - 25\right) - (2-10)\right]$$

$$= \frac{9}{2} + \frac{1}{2} + 2 + \frac{9}{2}$$

$$= \frac{23}{2}$$

Q. 13. Evaluate : $\displaystyle\int_1^3 \{|x-1| + |x-2| + |x-3|\}\,dx$

(*CBSE*, 2013)

Solution :

$1 < x < 3 \Rightarrow |x-1| = x-1$

$1 < x < 2 \Rightarrow |x-2| = -(x-2)$

$2 < x < 3 \Rightarrow |x-2| = x-2$

$1 < x < 3 \Rightarrow |x-3| = -(x-3)$

$$\therefore \; I = \int_1^3 \{|x-1| + |x-2| + |x-3|\}\,dx$$

$$= \int_1^3 |x-1|dx + \int_1^3 |x-2|dx + \int_1^3 |x-3|dx$$

$$= \int_1^3 |x-1|dx + \int_1^2 |x-2|dx + \int_2^3 |x-2|dx + \int_1^3 |x-3|dx$$

$$= \int_1^3 (x-1)\,dx - \int_1^2 (x-2)\,dx + \int_2^3 (x-2)\,dx - \int_1^3 (x-3)\,dx$$

$$= \left(\frac{x^2}{2} - x\right)_1^3 - \left(\frac{x^2}{2} - 2x\right)_1^2 + \left(\frac{x^2}{2} - 2x\right)_2^3 - \left(\frac{x^2}{2} - 3x\right)_1^3$$

$$= \left[\left(\frac{9}{2} - 3\right) - \left(\frac{1}{2} - 1\right)\right] - \left[2 - 2(2) - \left(\frac{1}{2} - 2\right)\right] + \left[\left(\frac{9}{2} - 6\right) - (2-4)\right] - \left[\left(\frac{9}{2} - 9\right) - \left(\frac{1}{2} - 3\right)\right]$$

$$= 2 + \frac{1}{2} + \frac{1}{2} + 2 = 5$$

Q. 14. Evaluate : $\int_0^4 \{|x| + |x-2| + |x-4|\}\,dx$ (*CBSE*, 2013)

Solution :

$0 < x < 4 \Rightarrow |x| = x$

$0 < x < 2 \Rightarrow |x-2| = -(x-2)$

$2 < x < 4 \Rightarrow |x-2| = x-2$

$0 < x < 4 \Rightarrow |x-4| = -(x-4)$

$$\therefore I = \int_0^4 \{|x| + |x-2| + |x-4|\}\,dx$$

$$= \int_0^4 |x|\,dx + \int_0^4 |x-2|\,dx + \int_0^4 |x-4|\,dx$$

$$= \int_0^4 |x|\,dx + \int_0^2 |x-2|\,dx + \int_0^4 |x-2|\,dx + \int_0^4 |x-4|\,dx$$

$$= \int_0^4 x\,dx - \int_0^2 (x-2)\,dx + \int_2^4 (x-2)\,dx - \int_0^4 (x-4)\,dx$$

$$= \left(\frac{x^2}{2}\right)_0^4 - \left(\frac{x^2}{2} - 2x\right)_0^2 + \left(\frac{x^2}{2} - 2x\right)_2^4 - \left(\frac{x^2}{2} - 4x\right)_0^4$$

$$= (8-0) - \{(2-4) - 0\} + \{(8-8) - (2-4)\} - \{(8-16) - 0\}$$

$$= 8 + 2 + 2 + 8$$

$$= 20$$

Long Answer Type Questions

Q. 1. Evaluate : $\int_0^1 (3x^2 + 1)\,dx$ as a limit of a sum. (*JAC*, 2014)

Solution :

Here

$a = 0$

$b = 1$

$nh = b - a = 1 - 0 = 1$

$f(x) = 3x^2 + 1$

$f(a) = f(0) = 1$

$f(a+h) = f(0+h) = f(h) = 3h^2 + 1$

$f(a+2h) = f(2h) = 3(2h)^2 + 1$

⋮

$f(a + \overline{n-1}h) = f(\overline{n-1}h) = 3(n-1)^2h^2 + 1$

$f(a+nh) = f(nh) = 3(n^2h^2) + 1$

Now $\int_0^1 (3x^2+1)\,dx$

$$= \lim_{h\to 0} h\,[f(a) + f(a+h) + h(a+2h) + \dots + f(a + \overline{n-1}h)] \text{ where } nh = 1$$

$$= \lim_{h\to 0} h\,[1 + (3h^2+1) + \{3(2h)^2 + 1\} + \dots + \{3(n-1)^2h^2 + 1\}] \text{ where } nh = 1$$

$$= \lim_{h\to 0} h\,[(n) + 3h^2\{1^2 + 2^2 + 3^2 + \dots + (h-1)^2\}] \text{ where } nh = 1$$

$$= \lim_{h\to 0} h\left[n + 3h^2\frac{n(n-1)(2n-1)}{6}\right] \text{ where } nh = 1$$

$$= \lim_{h\to 0}\left[nh + \frac{nh(nh-h)(2nh-h)}{2}\right] \text{ where } nh = 1$$

$$= 1 + \frac{1(1-0)(2.1-0)}{2}$$

$$= 1 + 1$$

$$= 2$$

Q. 2. Prove that :

$$\int_0^{\pi/2} \log\sin x\,dx = \int_0^{\pi/2} \log\cos x\,dx = -\frac{\pi}{2}\log 2$$

(*BSEB*, 2014)

Solution :

$$I = \int_0^{\pi/2} \log\sin x\,dx \quad \dots(1)$$

$$\Rightarrow \quad I = \int_0^{\pi/2} \log\sin\left(\frac{\pi}{2} - x\right)dx$$

$$\left(\because \int_0^a f(x)\,dx = \int_0^a f(a-x)\,dx\right)$$

$$\Rightarrow \quad I = \int_0^{\pi/2} \log\cos x\,dx \quad \dots(2)$$

Adding (1) and (2), we get

$$2I = \int_0^{\pi/2} (\log\sin x + \log\cos x)\,dx$$

$$= \int_0^{\pi/2} \log(\sin x\cos x)\,dx$$

$$= \int_0^{\pi/2} \log\left(\frac{2\sin x\cos x}{2}\right)dx$$

$$= \int_0^{\pi/2} \log\left(\frac{\sin 2x}{2}\right)dx$$

$$= \int_0^{\pi/2} \log\sin 2x\,dx - \int_0^{\pi/2} \log 2\,dx$$

Put $2x = t$

$\Rightarrow 2dx = dt$

$\Rightarrow dx = \frac{dt}{2}$

$$= \frac{1}{2}\int_0^{\pi} \log\sin t\,dt - \log 2\int_0^{\pi/2} dx$$

$$= \frac{1}{2}\int_0^{\pi} \log\sin x\,dx - \log 2\,(x)_0^{\pi/2}$$

$$= \frac{1}{2}2\int_0^{\pi/2} \log\sin x\,dx - \frac{\pi}{2}\log 2$$

$$= \int_0^{\pi/2} \log\sin x\,dx - \frac{\pi}{2}\log 2$$

$$= I - \frac{\pi}{2}\log 2$$

$$\Rightarrow \quad 2I - I = -\frac{\pi}{2}\log 2$$

$$\Rightarrow \quad I = -\frac{\pi}{2}\log 2$$

Q. 3. Evaluate : $\int_0^{\pi} \frac{x\tan x}{\sec x \operatorname{cosec} x}dx$ (*CBSE*, 2014)

Solution :

$$I = \int_0^{\pi} \frac{x\tan x}{\sec x \operatorname{cosec} x}dx \quad ...(1)$$

$$\Rightarrow \quad I = \int_0^{\pi} \frac{(\pi - x)\tan(\pi - x)}{\sec(\pi - x)\operatorname{cosec}(\pi - x)}dx$$

$$\left(\because \int_0^a f(x)dx = \int_0^a f(a-x)dx\right)$$

$$\Rightarrow \quad I = \int_0^{\pi} \frac{(\pi - x)(-\tan x)}{(-\sec x)(+\operatorname{cosec} x)}dx$$

$$\Rightarrow \quad I = \int_0^{\pi} \frac{(\pi - x)\tan x}{\sec x \operatorname{cosec} x}dx \quad ...(2)$$

Adding (1) and (2), we get

$$2I = \int_0^{\pi} \frac{\pi\tan x}{\sec x \operatorname{cosec} x}dx$$

$$\Rightarrow \quad I = \frac{\pi}{2}\int_0^{\pi} \frac{\sin x.\sin x}{\cos x \sec x}dx$$

$$= \frac{\pi}{2}\int_0^{\pi} \sin^2 x\,dx$$

$$= \frac{\pi}{4}\int_0^{\pi} (2\sin^2 x)dx$$

$$= \frac{\pi}{2}\int_0^{\pi} (1 - \cos 2x)dx$$

$$= \frac{\pi}{4}\left(x - \frac{\sin^2 x}{2}\right)_0^{\pi}$$

$$= \frac{\pi}{4}\left[(\pi - \sin\pi) - \left(0 - \frac{\sin 0}{2}\right)\right]$$

$$= \frac{\pi^2}{4}$$

Q. 4. Find the value :

$$\int_0^{\pi} \frac{x\tan x}{\sec x + \tan x}dx$$

[*CBSE Delhi*, 2010, *CBSE*, 14 (*Comptt.*)]

Solution :

Let $I = \int_0^{\pi} \frac{x\tan x}{\sec x + \tan x}dx \quad ...(1)$

$$I = \int_0^{\pi} \frac{(\pi - x)\tan(\pi - x)}{\sec(\pi - x) + \tan(\pi - x)}dx$$

$$= \int_0^{\pi} \frac{-(\pi - x)\tan x}{-\sec x - \tan x}$$

$$= \int_0^{\pi} \frac{(\pi - x)\tan x}{\sec x + \tan x}dx \quad ...(2)$$

Adding (1) and (2), we get

$$I + I = \int_0^{\pi} \frac{\pi\tan x}{\sec x + \tan x}dx = \pi\int_0^{\pi} \frac{\tan x}{\sec x + \tan x}dx$$

$$\Rightarrow \quad 2I = \pi\int_0^{\pi} \frac{\sin x}{1 + \sin x}dx$$

$$= \pi\int_0^{\pi}\left(1 - \frac{1}{1 + \sin x}\right)dx$$

$$= \pi\int_0^{\pi} 1dx - \pi\int_0^{\pi} \frac{1}{1 + \sin x}dx$$

$$= \pi(x)_0^{\pi} - \pi\int_0^{\pi} \frac{1 - \sin x}{\cos^2 x}dx$$

$$= \pi(\pi - 0) - \pi\int_0^{\pi}(\sec^2 x - \sec x\tan x)dx$$

$$\Rightarrow \quad 2I = \pi^2 - \pi\left[(\tan x - \sec x)_0^{x}\right]$$

$$\Rightarrow \quad 2I = \pi^2 - \pi(0 + 1 - 0 + 1)$$

$$\Rightarrow \quad I = \frac{\pi^2}{2} - \frac{2\pi}{2} = \pi\left(\frac{\pi}{2} - 1\right)$$

$$\therefore \quad I = \pi\left(\frac{\pi}{2} - 1\right)$$

Q. 5. Evaluate : $\int_0^{\pi/4} \frac{\sin 2\theta}{\sin^4\theta + \cos^4\theta}d\theta$

[*CBSE*, 2013 (*Comptt.*)]

Solution :

$$I = \int_0^{\pi/4} \frac{\sin 2\theta}{\sin^4\theta + \cos^4\theta}d\theta$$

$$= \int_0^{\pi/4} \frac{2\sin\theta\cos\theta}{\sin^4\theta + \cos^4\theta}d\theta$$

$$= \int_0^{\pi/4} \frac{2\sec^2\theta\tan\theta}{\tan^4\theta + 1}d\theta$$

(dividing the numerator and denominator in the integrand by $\cos^4\theta$)

Put $\tan^2\theta = t$

$\therefore \quad 2\tan\theta\sec^2\theta\,d\theta = dt$

when $\theta = 0$, $t = \tan^2 0 = 0$

When $\theta = \frac{\pi}{4}$, $t = \tan^2\frac{\pi}{4} = 1$

$$= \int_0^1 \frac{dt}{t^2 + 1}$$

$$= (\tan^{-1} t)_0^1 = \tan^{-1}(1) - \tan^{-1}(0)$$

$$= \frac{\pi}{4} - 0 = \frac{\pi}{4}$$

Q. 6. Evaluate the following integral :

$$\int_0^{\pi} \frac{x\sin x}{1 + \cos^2 x}dx$$

(*USEB*, 2009, *CBSE*, *Outside Delhi*, 2012, 13 *BSER*, 2014, *AI CBSE*, 2013)

Solution :

Let $I = \int_0^{\pi} \frac{x\sin x}{1 + \cos^2 x}dx \quad ...(i)$

then $I = \int_0^{\pi} \frac{(\pi - x)\sin(\pi - x)}{1 + \cos^2(\pi - x)}dx$

$$= \int_0^{\pi} \frac{(\pi - x)\sin x}{1 + \cos^2 x}dx \quad ...(ii)$$

Adding (i) and (ii), we get

$$I + I = \int_0^{\pi} \frac{x\sin x}{1+\cos^2 x}dx + \int_0^{\pi} \frac{(\pi - x)\sin x}{1+\cos^2 x}dx$$

$$\Rightarrow \quad 2I = \int_0^{\pi} \frac{\pi \sin x}{1+\cos^2 x}dx$$

Taking $\cos x = t$,

$\Rightarrow \quad \sin x\, dx = -dx$

if $x = \pi$, $t = -1$ and if $x = 0$, $t = 1$

$$\therefore \quad 2I = \pi\int_1^{-1} \frac{-dt}{1+t^2}$$

$$\Rightarrow \quad 2I = -\pi\int_1^{-1} \frac{-dt}{1+t^2}$$

$$\therefore \quad I = \frac{\pi}{2}\int_{-1}^{1} \frac{-dt}{1+t^2}$$

$$= \frac{\pi}{2}(\tan^{-1} t)_{-1}^{1}$$

$$= \frac{\pi}{2}\left(\frac{\pi}{4} + \frac{3\pi}{4}\right)$$

$$= \frac{\pi}{2}\left(\frac{4\pi}{4}\right) = \frac{\pi^2}{2}$$

Q. 7. Show that :

$$\int_0^{\pi/2} \frac{\sin^2 x}{\sin x + \cos x}dx = \frac{1}{\sqrt{2}}\log(\sqrt{2}+1)$$

[*AI CBSE*, 2014 (*Comptt.*)]

Solution :

$$I = \int_0^{\pi/2} \frac{\sin^2 x}{\sin x + \cos x}dx \quad ...(1)$$

$$\Rightarrow \quad I = \int_0^{\pi/2} \frac{\sin^2\left(\frac{\pi}{2} - x\right)}{\sin\left(\frac{\pi}{2} - x\right) + \cos\left(\frac{\pi}{2} - x\right)}dx$$

$$\left(\because \int_0^a f(x)dx = \int_0^a f(a-x)\, dx\right)$$

$$\Rightarrow \quad I = \int_0^{\pi/2} \frac{\cos^2 x}{\cos x + \sin x}dx$$

$$\Rightarrow \quad I = \int_0^{\pi/2} \frac{\cos^2 x}{\sin x + \sin x}dx \quad ...(2)$$

Adding (1) and (2), we get

$$\Rightarrow \quad 2I = \int_0^{\pi/2} \frac{\sin^2 x + \cos^2 x}{\sin x + \cos x}dx$$

$$= \int_0^{\pi/2} \frac{1}{\sin x + \cos x}dx$$

$$= \int_0^{\pi/2} \frac{1}{\sqrt{2}\left(\sin x \sin\frac{\pi}{4} + \cos x \cos\frac{\pi}{4}\right)}dx$$

$$= \frac{1}{\sqrt{2}}\int_0^{\pi/2} \frac{1}{\cos\left(x - \frac{\pi}{4}\right)}dx$$

$$= \frac{1}{\sqrt{2}}\int_0^{\pi/2} \sec\left(x - \frac{\pi}{4}\right)dx$$

Put $x - \frac{\pi}{4} = t$

$\therefore \quad dx = dt$

$$= \frac{1}{\sqrt{2}}\int_{-\pi/4}^{\pi/4} \sec t\, dt$$

$$= \frac{2}{\sqrt{2}}\int_0^{\pi/4} \sec t\, dt$$

$$= \sqrt{2}\,[\log(\sec t + \tan t)]_0^{\pi/4}$$

$$= \sqrt{2}\left[\log\left(\sec\frac{\pi}{4} + \tan\frac{\pi}{4}\right) - \log(\sec 0 + \tan 0)\right]$$

$$= \sqrt{2}\,[\log(\sqrt{2}+1) - \log(1+0)]$$

$$= \sqrt{2}\,\log(\sqrt{2}+1)$$

$$\Rightarrow \quad I = \frac{1}{\sqrt{2}}\log(\sqrt{2}+1)$$

Q. 8. Evaluate :

$$I = \int_0^{\pi} \frac{x\,dx}{1+\sin x}$$

(*JAC*, 2014)

Solution :

Given $\quad I = \int_0^{\pi} \frac{x\,dx}{1+\sin x}$

then $\quad I = \int_0^{\pi} \frac{(\pi - x)dx}{1+\sin(\pi - x)}$

$$\left[\because \int_0^a f(x)dx = \int_0^a f(a-x)dx\right]$$

$$\Rightarrow \quad I = \int_0^{\pi} \frac{(\pi - x)dx}{1+\sin x} \quad ...(2)$$

Adding (1) and (2), we get

$$2I = \pi\int_0^{\pi} \frac{1}{1+\sin x}dx$$

$$= \pi\int_0^{\pi} \frac{1}{1+\frac{2\tan x/2}{1+\tan^2 x/2}}dx$$

$$= 2\pi\int_0^{2\pi/2} \frac{\sec^2 x/2}{1+\tan^2\frac{x}{2} + 2\tan\frac{x}{2}}dx$$

Let $\quad \tan\frac{x}{2} = 1$

$\Rightarrow \quad \sec^2\frac{x}{2}\, dx = 2dt$

therefore, $\quad 2I = (2\pi)\int_0^1 \frac{1}{(1+t^2+2t)}2dt$

$$\Rightarrow \quad I = 2\pi\int_0^1 \frac{1}{(t+1)^2}dt$$

$$\therefore \quad I = 2\pi \times \left[-\frac{1}{(t+1)}\right]_0^1$$

$$= 2\pi\left[-\frac{1}{2}+\frac{1}{1}\right]$$

$$= 2\pi\times\frac{1}{2} = \pi.$$

Q. 9. Evaluate :

$$\int_0^\pi \frac{x\,dx}{a^2\cos^2 x + b^2\sin^2 x}.$$

(*CBSE, AI*, 2009, *USEB*, 2013)

Solution :

$$\text{Let I} = \int_0^\pi \frac{x\,dx}{a^2\cos^2 x+b^2\sin^2 x}$$

$$= \int_0^\pi \frac{(\pi-x)\,dx}{a^2\cos^2(\pi-x)+b^2\sin^2(\pi-x)}$$

$$= \pi\int_0^\pi \frac{dx}{a^2\cos^2 x+b^2\sin^2 x} - \int_0^\pi \frac{x\,dx}{a^2\cos^2 x+b^2\sin^2 x}$$

$$\Rightarrow \quad \text{I} = \pi\int_0^\pi \frac{dx}{a^2\cos^2 x+b^2\sin^2 x} - \text{I}$$

$$\Rightarrow 2\text{I} = \pi\int_0^\pi \frac{dx}{a^2\cos^2 x+b^2\sin^2 x}$$

$$\Rightarrow 2\text{I} = 2\pi\int_0^{\pi/2} \frac{dx}{a^2\cos^2 x+b^2\sin^2 x}$$

$$\therefore \quad \text{I} = \pi\int_0^{\pi/2} \frac{\sec^2 x\,dx}{a^2+b^2\tan^2 x}$$

[Dividing by $\cos^2 x$ in numerator & denominator,
Putting $\tan x = t \Rightarrow \sec^2 x\,dx = dt$
when $x = 0 \Rightarrow t = 0$
and $x = \pi/2 \Rightarrow t = \infty$]

$$\therefore \quad \text{I} = \pi\int_0^\infty \frac{dt}{a^2+b^2t^2}$$

$$= \frac{\pi}{b^2}\int_0^\infty \frac{dt}{\frac{a^2}{b^2}+t^2}$$

$$= \frac{\pi}{b^2}\times\frac{1}{a/b}\left[\tan^{-1}\frac{t}{a/b}\right]_0^\infty$$

$$= \frac{\pi}{ab}[\tan^{-1}(\infty)-\tan^{-1}(0)]$$

$$= \frac{\pi}{ab}\times\frac{\pi}{2} = \frac{\pi^2}{2ab}$$

Q. 10. Evaluate :

$$\int_0^{\pi/4} \frac{\sin x+\cos x}{9+16\sin 2x}dx$$ [*CBSE*, 2014 (*Comptt.*)]

Solution :

$$\text{I} = \int_0^{\pi/4}\frac{\sin x+\cos x}{9+16\sin 2x}dx$$

Put $\sin x - \cos x = t$

$\therefore$ $(\cos x + \sin x)\,dx = dt$;

Squaring, we get

$\sin^2 x + \cos^2 x - 2\sin x\cos x = t^2$

$\Rightarrow$ $1 - \sin 2x = t$

$\Rightarrow$ $\sin 2x = 1 - t^2$;

when $x = 0$, $t = \sin 0 - \cos 0 = 0 - 1 = -1$

when $x = \frac{\pi}{4}$, $t = \sin\frac{\pi}{4} - \cos\frac{\pi}{4} = \frac{1}{\sqrt2}-\frac{1}{\sqrt2} = 0$

$$\therefore \quad \text{I} = \int_{-1}^0 \frac{dt}{9+16(1-t^2)}$$

$$= \int_{-1}^0 \frac{dt}{25-16t^2}$$

$$= \frac{1}{16}\int_{-1}^0 \frac{dt}{\frac{25}{16}-t^2}$$

$$= \frac{1}{16}\int_{-1}^0 \frac{dt}{\left(\frac{5}{4}\right)^2-t^2}$$

$$= \frac{1}{16}\cdot\frac{1}{2\cdot\left(\frac{5}{4}\right)}\left[\log\frac{\frac{5}{4}+t}{\frac{5}{4}-t}\right]_{-1}^0$$

$$= \frac{1}{40}\left[\log\left(\frac{5+4t}{5-4t}\right)\right]_{-1}^0$$

$$= \frac{1}{40}\left(\log 1 - \log\frac{1}{9}\right)$$

$$= \frac{1}{40}\log 9$$

Q. 11. Prove that :

$$\int_0^{\pi/4}(\sqrt{\tan x}+\sqrt{\cot x})\,dx = \sqrt{2}.\frac{\pi}{2}$$

(*CBSE Delhi*, 2012)

Solution :

$$\text{Let I} = \int_0^{\pi/4}(\sqrt{\tan x}+\sqrt{\cot x})\,dx$$

$$= \int_0^{\pi/4}\left(\frac{\sqrt{\sin x}}{\sqrt{\cos x}}+\frac{\sqrt{\cos x}}{\sqrt{\sin x}}\right)dx$$

$$= \int_0^{\pi/4}\left(\frac{\sin x+\cos x}{\sqrt{\sin x\cos x}}\right)dx$$

$$= \sqrt2\int_0^{\pi/4}\frac{\sin x+\cos x}{\sqrt{2\sin x\cos x}}dx$$

$$= \sqrt2\int_0^{\pi/4}\frac{\sin x+\cos x}{\sqrt{1-(1-2\sin x\cos x)}}dx$$

$$= \sqrt2\int_0^{\pi/4}\frac{\sin x+\cos x}{\sqrt{1-(\sin^2 x+\cos^2 x-2\sin x\cos x)}}dx$$

$$= \sqrt2\int_0^{\pi/4}\frac{\sin x+\cos x}{\sqrt{1-(\sin x-\cos x)^2}}dx$$

Putting $\sin x - \cos x = t$,

then $(\cos x + \sin x)\,dx = dt$

Also, when $x = 0$, $t = 0 - 1 = -1$

and when $x = \frac{\pi}{2}$, $t = \frac{1}{\sqrt2} - \frac{1}{\sqrt2} = 0$

$$\therefore \quad \text{I} = \sqrt2\int_{-1}^0 \frac{dt}{\sqrt{1-t^2}}$$

$$= \sqrt{2}[\sin^{-1} t]_{-1}^{0}$$
$$= \sqrt{2}[\sin^{-1} 0 - \sin^{-1}(-1)]$$
$$= \sqrt{2}[0 + \sin^{-1}(1)]$$
$$= \sqrt{2}.\frac{\pi}{2}$$

Q. 12. Evaluate :

$\int_1^3 (2x^2 + 5x)\,dx$ **as a limit of a sum.**

(*CBSE Delhi*, 2012)

Solution :

We know that

$$\int_a^b f(x)dx = \lim_{h\to\infty} h[f(a) + f(a+h) + f(a+2h) + ... + f\{a + (n-1)h\}]$$

where $h = \frac{b-a}{n}$

Here $a = 1$, $b = 3$, $f(x) = 2x^2 + 5x$

and $h = \frac{3-1}{n} = \frac{2}{n}$ or $nh = 2$

$$\therefore \int_1^3 (2x^2+5x)\,dx = \lim_{h\to\infty} h[f(1) + f(1+h) + f(1+2h) + ... + f\{1 + (n-1)h\}]$$

$$= \lim_{h\to 0} h[2.(1)^2 + 5] + \{2(1+h)^2 + 5(1+h)\} + \{2(1+2h)^2 + 5(1+2h)\} + ... + \{2.(1+(n-1)h)^2 + 5(1+n-1)h)\}]$$

$$= \lim_{h\to 0} h[7 + \{7 + 9h + 2h^2\} + \{7 + 18h + 8h^2\} + ... + \{7 + 9(n-1)h + 2(n-1)2h^2]$$

$$= \lim_{h\to 0} h[7n + 9h\{1 + 2 + ... + (n-1)\} + 2h^2\{1^2 + 2^2 + ... + (n-1)^2\}$$

$$= \lim_{h\to 0}\left[7n + 9h\left\{\frac{n(n-1)}{2}\right\} + 2h^2\left\{\frac{n(n-1)(2n-1)}{6}\right\}\right]$$

$$= \lim_{h\to 0} h\left[7nh + 9\times\frac{nh(nh-h)}{2} + \frac{nh(nh-h)(2nh-h)}{3}\right]$$

$$= 7\times 2 + \frac{9}{2}\times 2(2-0) + \frac{2(2-0)(4-0)}{3}$$

$$= 14 + 18 + \frac{16}{3}$$

$$= \frac{112}{3}$$

Q. 13. Evaluate : $\int_0^{\pi/4} \sin^2 x\,dt$ (*BSEB*, 2015)

Solution :

$$\int_0^{\pi/4} \sin^2 x\,dx = \frac{1}{2}\int_0^{\pi/4}(1-\cos 2x)\,dx = \left[x - \frac{\sin 2x}{2}\right]_0^{\pi/4}$$

$$= \frac{1}{2}\left[\left(\frac{\pi}{4} - \frac{\sin \pi/2}{2}\right) - \left(0 - \frac{\sin 0}{2}\right)\right]$$

$$= \frac{1}{2}\left(\frac{\pi}{4} - \frac{1}{2}\right)$$

$$= \frac{\pi}{8} - \frac{1}{4}$$

Q. 14. Prove that : $\int_0^{\pi/2} \frac{\sqrt{\sin x}}{\sqrt{\sin x} + \sqrt{\cos x}}\,dx = \frac{\pi}{4}$

(*JAC*, 2015)

Solution :

Let
$$I = \int_0^{\pi/2} \frac{\sqrt{\sin x}}{\sqrt{\sin x} + \sqrt{\cos x}}\,dx \quad ...(1)$$

$$= \int_0^{\pi/2} \frac{\sqrt{\sin\left(\frac{\pi}{2} - x\right)}}{\sqrt{\sin\left(\frac{\pi}{2} - x\right)} + \sqrt{\cos\left(\frac{\pi}{2} - x\right)}}\,dx$$

$$= \int_0^{\pi/2} \frac{\sqrt{\cos x}}{\sqrt{\cos x} + \sqrt{\sin x}}\,dx \quad ...(2)$$

Adding equations (1) and (2),

$$2I = \int_0^{\pi/2} \frac{\sqrt{\sin x}}{\sqrt{\sin x} + \sqrt{\cos x}}\,dx + \int_0^{\pi/2} \frac{\sqrt{\cos x}}{\sqrt{\sin x} + \sqrt{\cos x}}\,dx$$

$$= \int_0^{\pi/2} \frac{\sqrt{\sin x} + \sqrt{\cos x}}{\sqrt{\sin x} + \sqrt{\cos x}}\,dx$$

$$= \int_0^{\pi/2} 1.dx = [x]_0^{\pi/2} = \frac{\pi}{2}$$

$$\therefore I = \frac{\pi}{4}$$

Q. 15. Evaluate : $\int_1^3 (x + x)^2\,dx$ **as a limit of a sum.**

(*JAC*, 2015)

Solution : $\int_1^3 (x+x)^2\,dx$

$$= \lim_{h\to 0} h\,[f(1) + f(1+h) + ... + f(1 + (n-1)\,h)]$$

$$= \lim_{h\to 0} h\,[(1^2+1) + [(1+h)^2 + (1+h)\} + ... \{(1+(n-1)\,h)^2 + (1 + (n-1)\,h)\}]$$

$$= \lim_{h\to 0} h\,[\{1^2 + (1+h)^2 + (1+2h)^2 + ... + (1 + (n-1)\,h^2)\} + \{1 + (1+h) + (1+2h) + ... + (1 + (n-1)\,h)\}]$$

$$= \lim_{h\to 0} h\left[n + 2h\frac{n(n-1)}{2} + h^2\frac{n(n-1)(2n-1)}{6} + n + h\frac{n(n-1)}{2}\right]$$

$$= \lim_{h\to 0} h\left[2n + 3h\frac{n(n-1)}{2} + h^2\frac{n(n-1)(2n-1)}{6}\right]$$

$$= \lim_{h\to 0} \frac{2}{n}\left[2n + \frac{6}{n}\frac{n(n-1)}{2} + \frac{4}{n^2}\frac{n(n-1)(2n-1)}{6}\right]$$

$$= \lim_{h\to 0}\left[4 + 6\left(\frac{n-1}{n}\right) + \frac{4}{3}\frac{(n-1)(2n-1)}{n^2}\right]$$

$$= \lim_{h\to 0}\left[4 + 6\left(1 - \frac{1}{n}\right) + \frac{4}{3}\left(1 - \frac{1}{n}\right)\left(2 - \frac{1}{n}\right)\right]$$

$$= 4 + 6\,(1-0) + \frac{4}{3}\,(1-0)\,(2-0)$$

$$= 4 + 6 + \frac{8}{3}$$

$$= \frac{38}{3}$$

Q. 16. Evaluate : $\int_0^{\pi/4} \sin 2x\, dx.$ (*USEB*, 2015)

Solution : $\int_0^{\pi/4} \sin 2x\, dx$

$$= \left[-\frac{\cos 2x}{2}\right]_0^{\pi/4}$$

$$= \frac{-1}{2}[\cos \pi/2 - \cos 0]$$

$$= \frac{-1}{2}[0-1]$$

$$= \frac{1}{2}$$

NCERT QUESTIONS

Q. 1. Evaluate :

$$\int_0^{\pi/2} 2\sin x \cos x \tan^{-1}(\sin x)\,dx.$$

(*CBSE*, 2011)

Solution :

$$\int_0^{\pi/2} 2\sin x \cos x \tan^{-1}(\sin x)\,dx$$

$\left\{\begin{array}{l}\text{Let } \sin x = t \\ \Rightarrow \cos x\, dx = dt\end{array}\right.$

Also, $\sin 0 = 0,\ \sin\frac{\pi}{2} = 1$

$$= \int_0^1 \underset{(2)}{2t} \underset{(1)}{\tan^{-1}(t)}\, dt$$

$$= 2\left[\left\{\tan^{-1}(t) \times \frac{t^2}{2}\right\}_0^1 - \int_0^1 \frac{1}{1+t^2} \times \frac{t^2}{2} dt\right]$$

$$= 2\left[\left(\frac{1}{2}\times\frac{\pi}{4}\right) - \frac{1}{2}\left\{\int_0^1 \frac{t^2+1}{t^2+1}dt - \int_0^1 \frac{1}{t^2+1}dt\right\}\right]$$

$$= 2\left[\frac{\pi}{8} - \frac{1}{2}\left\{(t)_0^1 - (\tan^{-1} t)_0^1\right\}\right]$$

$$= 2\left[\frac{\pi}{8} - \frac{1}{2}\left\{1 - \frac{\pi}{4}\right\}\right]$$

$$= 2\left[\frac{\pi}{8} - \frac{1}{2} + \frac{\pi}{8}\right]$$

$$= 2\left[\frac{\pi}{4} - \frac{1}{2}\right]$$

$$= \frac{\pi}{2} - 1$$

Q. 2. Evaluate :

$$\int_0^{\pi/2} \frac{x\sin x\cos x}{\sin^4 x + \cos^4 x} dx.$$

(*CBSE*, 2011,14)

Solution :

$$I = \int_0^{\pi/2} \frac{x\sin x\cos x}{\sin^4 x + \cos^4 x} dx \quad \ldots(1)$$

$$I = 2\int_0^{\pi/2} \frac{\left(\frac{\pi}{2}-x\right)\sin\left(\frac{\pi}{2}-x\right)\cos\left(\frac{\pi}{2}-x\right)}{\sin^4\left(\frac{\pi}{2}-x\right)+\cos^4\left(\frac{\pi}{2}-x\right)} dx$$

$$= \frac{\pi}{2}\int_0^{\pi/2} \frac{\cos x \sin x}{\cos^4 x + \sin^4 x} dx - \int_0^{\pi/2} \frac{x\cos x\sin x}{\cos^4 x+\sin^4 x} dx \quad \ldots(2)$$

Adding (1) and (2), we get

$$2I = \frac{\pi}{2}\int_0^{\pi/2} \frac{\sin x\cos x}{\sin^4 x + \cos^4 x} dx$$

$$\Rightarrow \quad I = \frac{\pi}{4}\int_0^{\pi/2} \frac{\sin x\cos x}{\sin^4 x + \cos^4 x} dx$$

$$\Rightarrow \quad I = \frac{\pi}{8}\int_0^{\pi/2} \frac{\sin 2x}{\sin^4 x + \cos^4 x} dx$$

$$\Rightarrow \quad I = \frac{\pi}{8}\int_0^{\pi/2} \frac{\sin 2x}{\sin^4 x + (1-\sin^2 x)^2} dx,$$

$\left\{\begin{array}{l}\text{Let } \sin^2 x = t \\ \Rightarrow \sin 2x\, dx = dt \\ \text{when } x = 0 \Rightarrow t = \sin^2 0 = 0 \\ \text{and } x = \frac{\pi}{2} \Rightarrow t = \sin^2\frac{\pi}{2} = 1\end{array}\right.$

$$\Rightarrow \quad I = \frac{\pi}{8}\int_0^1 \frac{dt}{t^2+(1-t)^2}$$

$$\Rightarrow \quad I = \frac{\pi}{8}\int_0^1 \frac{dt}{2t^2-2t+1}$$

$$\Rightarrow \quad I = \frac{\pi}{16}\int_0^1 \frac{dt}{t^2-t+\frac{1}{2}}$$

$$\Rightarrow \quad I = \frac{\pi}{16}\int_0^1 \frac{dt}{t^2-t+\frac{1}{2}+\frac{1}{4}-\frac{1}{4}}$$

$$\Rightarrow \quad I = \frac{\pi}{16}\int_0^1 \frac{dt}{\left(t-\frac{1}{2}\right)^2+\frac{1}{4}}$$

$$\Rightarrow \quad I = \frac{\pi}{16}\times\frac{1}{1}\times 2\left[\tan^{-1}\left(\frac{t-\frac{1}{2}}{\frac{1}{2}}\right)\right]_0^1$$

$$\Rightarrow \quad I = \frac{\pi}{8}\left[\tan^{-1}(2t-1)\right]_0^1$$

$$\Rightarrow \quad I = \frac{\pi}{8}\left[\tan^{-1}1 - \tan^{-1}(-1)\right]$$

$$\therefore \quad I = \frac{\pi}{8}\left(\frac{\pi}{4}+\frac{\pi}{4}\right)$$

$$= \frac{\pi}{8}\times\frac{\pi}{2}$$

$$= \frac{\pi^2}{16}$$

Q. 3. Evaluate :

$$\int_0^{\pi/2} \frac{x\sin x}{1+\cos x} dx$$

(*CBSE Outside Delhi*, 2011)

$$I = \int_0^{\pi/2} \frac{x\sin x}{1+\cos x} dx$$

$$= \int_0^{\pi/2} \frac{x}{1+2\cos^2\frac{x}{2}-1}dx + \int_0^{\pi/2} \frac{2\sin\frac{x}{2}\cos\frac{x}{2}}{1+2\cos^2\frac{x}{2}-1}dx$$

$$= \frac{1}{2}\int_0^{\pi/2} x\sec^2\frac{x}{2}dx + \int_0^{\pi/2}\tan\frac{x}{2}\,dx$$

$$= \frac{1}{2}\left[\left(2x\tan\frac{x}{2}\right)_0^{\frac{\pi}{2}} - \int_0^{\frac{\pi}{2}} 2\tan\frac{x}{2}dx\right] + -\int_0^{\frac{\pi}{2}}\tan\frac{x}{2}dx$$

$$= \left(x\tan\frac{x}{2}\right)_0^{\pi/2}$$

$$= \frac{\pi}{2}\tan\frac{\pi}{4} - 0 = \frac{\pi}{4}$$

Q. 4. Evaluate :

$\int_0^{\pi/4} \log(1+\tan x)\,dx$ *(CBSE Outside Delhi,* 2011)

Solution :

$$I = \int_0^{\pi/4} \log(1+\tan x)dx \qquad \ldots(1)$$

$$\Rightarrow \quad I = \int_0^{\pi/4} \log\left(1+\tan\left(\frac{\pi}{4}-x\right)\right)dx$$

$$\Rightarrow \quad I = \int_0^{\pi/4} \log\left(1+\left\{\frac{\tan\frac{\pi}{4}-\tan x}{1-\tan\frac{\pi}{4}\tan x}\right\}\right)dx$$

$$\Rightarrow \quad I = \int_0^{\pi/4} \log\left[1+\frac{1-\tan x}{1+\tan x}\right]dx$$

$$\Rightarrow \quad I = \int_0^{\pi/4} \log\left(\frac{2}{1+\tan x}\right)dx$$

$$\Rightarrow \quad I = \int_0^{\pi/4} \log 2\,dx - \int_0^{\pi/4} \log(1+\tan x)\,dx$$

$$\Rightarrow \quad I = (\log 2)\,[x]_0^{\pi/4} - I$$

$$\Rightarrow \quad 2I = \frac{\pi}{4}\log 2$$

$$\therefore \quad I = \frac{\pi}{8}\log 2 \qquad \text{[from (1)]}$$

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