4 DETERMINANTS

IMPORTANT FORMULAE

Determinants

- Determinant of order 2 : $\begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} = a_1b_2 - a_2b_1$
- Determinant of order 3 :

$$\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = a_1 \begin{vmatrix} b_2 & b_3 \\ c_2 & c_3 \end{vmatrix} - a_2 \begin{vmatrix} b_1 & b_3 \\ c_1 & c_3 \end{vmatrix} + a_3 \begin{vmatrix} b_1 & b_2 \\ c_1 & c_2 \end{vmatrix}$$

$$= a_1(b_2c_3 - b_3c_2) - a_2(b_1c_3 - b_3c_1) + a_3(b_1c_2 - b_2c_1)$$

Minors and Co-factors :

- The minor M_{ij} of an element a_{ij} is the value of the determinant obtained by deleting the ith row and jth column of given determinant,
  and co-factor of a_{ij}, $C_{ij} = (-1)^{i+j} M_{ij}$

  Minor of a_2 in given above det. $= \begin{vmatrix} b_1 & b_3 \\ c_1 & c_3 \end{vmatrix}$

  $\because$ a_2 belongs in 1st row and 2nd column

  $\therefore$ Co-factor of $a_2 = (-1)^{1+2} \begin{vmatrix} b_1 & b_3 \\ c_1 & c_3 \end{vmatrix}$

  $= \begin{vmatrix} b_1 & b_3 \\ c_1 & c_3 \end{vmatrix} = -(b_1c_3 - b_3c_1)$

  Minor of $b_1 = \begin{vmatrix} a_2 & a_3 \\ c_2 & c_3 \end{vmatrix}$ and Co-factor of $b_1 = \begin{vmatrix} a_2 & a_3 \\ c_2 & c_3 \end{vmatrix}$

  Minor of $b_2 = \begin{vmatrix} a_1 & a_3 \\ c_1 & c_3 \end{vmatrix}$ and Co-factor of $b_2 = \begin{vmatrix} a_1 & a_3 \\ c_1 & c_3 \end{vmatrix}$

  Co-factors of $a_1, a_2, a_3, b_1, b_2, b_3, c_1, c_2, c_3$ are denoted by the capital letters $A_1, A_2, A_3, B_1, B_2, B_3, C_1, C_2, C_3$ respectively.

Properties of Determinants :

- If rows be changed into columns and columns into rows, the determinant remains unaltered.
- If two rows or two columns in a determinant are identical, the value of the determinant is equal to zero.
- If all the elements of any row or column be multiplied by a non-zero real numbers k, then the value of the new determinant is k times, the value of the original determinant.
- If each entry in a row or column of a determinant is written as the sum of two or more terms, then the determinant can be written as the sum of two or more determinants.
- If each entry of one row or column of a determinant is multiplied by a real number k and the resulting product is added to the corresponding entry in another row or column in the determinant then the resulting determinant is equal to the original determinant.
- If each entry in any row or column of a determinant is 0, then the value of the determinant is equal to 0.
- If any two adjacent rows or columns of a determinant are interchanged, the resulting determinant is the negative of original determinant.

Area of Triangle :

- If the vertices of ΔABC are $A(x_1, y_1)$, $B(x_2, y_2)$ and $C(x_3, y_3)$, then

$$\text{area } \Delta ABC = \frac{1}{2}\begin{vmatrix} x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \\ 1 & 1 & 1 \end{vmatrix} = \frac{1}{2}\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}$$

- If the points A, B and C are collinear, then the area of $\Delta ABC = 0$
- If the elements of one row or column and corresponding co-factors of the other row or column are multiplied and added, then this sum is 0. For example :

$$a_{11}A_{21} + a_{12}A_{22} + a_{13}A_{23} = 0$$

- If the matrix $A = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$, then

  adj. $A = \begin{vmatrix} A_{11} & A_{21} & A_{31} \\ A_{12} & A_{22} & A_{32} \\ A_{13} & A_{23} & A_{33} \end{vmatrix}$, where A_{ij} in the co-factor of a_{ij}.

- A (adj A) = (adj A) A = $|A|$ I, where A is the square matrix of order n.
- Any square matrix A possesses its inverse if and only if A is invertible.
- $A^{-1} = \frac{1}{|A|}$ (adj A)

  If
  $$a_1x + b_1y + c_1z = d_1$$
  $$a_2x + b_2y + c_2z = d_2$$
  $$a_3x + b_3y + c_3z = d_3$$
  then this system can be written in the form of AX = B.

  where $A = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$, $X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ and $B = \begin{bmatrix} d_1 \\ d_2 \\ d_3 \end{bmatrix}$

Multiple Choice Questions

1. If $\Delta = \begin{vmatrix} 1 & a & b+c \\ 1 & b & c+a \\ 1 & c & a+b \end{vmatrix}$, then $\Delta =$ *(BSEB, 2011)*

   (a) abc (b) 0
   (c) $a + b + c$ (d) none of the above

2. If $\Delta = \begin{vmatrix} 10 & 2 \\ 30 & 6 \end{vmatrix}$, then $\Delta =$ (*BSEB*, 2011)

(a) 0 (b) 10 (c) 12 (d) 60

3. If $\begin{vmatrix} 1-x & 2 \\ 18 & 6 \end{vmatrix} = \begin{vmatrix} 6 & 2 \\ 18 & 6 \end{vmatrix}$, then the value of x is : (*BSEB*, 2012)

(a) ± 6 (b) 6 (c) – 5 (d) 7

4. If $A = \begin{vmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{vmatrix}$, then adj A is : (*BSEB*, 2012)

(a) $\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$ (b) $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

(c) $\begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$ (d) $\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$

5. If w is a non-real root of the equation $x^3 - 1 = 0$, then

$\begin{vmatrix} 1 & w & w^2 \\ w & w^2 & 1 \\ w^2 & 1 & w \end{vmatrix} =$ (*BSEB*, 2012)

(a) 0 (b) 1 (c) w (d) w^2

6. The value of the determinant $\begin{vmatrix} 3 & 1 & 7 \\ 5 & 0 & 2 \\ 2 & 5 & 3 \end{vmatrix}$ is : (*BSEB*, 2013)

(a) 124 (b) 125 (c) 134 (d) 144

7. The value of $\begin{vmatrix} 4a & 1 & 6 \\ 3a & 7 & 4 \\ 10 & 2 & 1 \end{vmatrix}$ is :

(a) 1 (b) 2 (c) 4 (d) 0

8. The value of $\begin{vmatrix} a-b & m-n & x-y \\ b-c & n-p & y-z \\ c-a & p-m & z-x \end{vmatrix}$ is :

(a) 0 (b) $a + b + c$
(c) $m + n + p$ (d) $x - y + z$

9. The co-factors of the first column elements in the determinant $\begin{vmatrix} 5 & 20 \\ 3 & -1 \end{vmatrix}$ are :

(a) – 1, 3 (b) – 1, – 3 (c) – 1, 20 (d) – 1, – 20

10. Which of the following determinant is equal to the determinant $\begin{vmatrix} 1 & 0 & 2 \\ 3 & -2 & -1 \\ 1 & 1 & 2 \end{vmatrix}$?

(a) $-\begin{vmatrix} 2 & 5 & 4 \\ 3 & -2 & -1 \\ 1 & 1 & 2 \end{vmatrix}$ (b) $\begin{vmatrix} 1 & 3 & 2 \\ 2 & -1 & 4 \\ 0 & 2 & 5 \end{vmatrix}$

(c) $\begin{vmatrix} 2 & -1 & 4 \\ 0 & -2 & 5 \\ 1 & 3 & 2 \end{vmatrix}$ (d) $-\begin{vmatrix} 2 & 0 & 1 \\ -1 & -2 & 3 \\ 4 & 5 & 2 \end{vmatrix}$

11. Find the value of $\begin{vmatrix} 1 & x & x^2 \\ 1 & y & y^2 \\ 1 & z & z^2 \end{vmatrix} =$ (*BSEB*, 2015)

(a) $(x - y)(y + z)(z + x)$ (b) $(x + y)(y - z)(z - x)$
(c) $(x - y)(y - z)(z + x)$ (d) $(x - y)(y - z)(z - x)$

12. Find the value of $\begin{vmatrix} 3 & 4 & 5 \\ 0 & 2 & 3 \\ 0 & 0 & 7 \end{vmatrix} =$ (*BSEB*, 2015)

(a) 40 (b) 50 (c) 42 (d) 15

Ans. 1. (b), 2. (a), 3. (c), 4. (c), 5. (a), 6. (c), 7. (d), 8. (a), 9. (d), 10. (c), 11. (d), 12. (c).

Very Short Answer Type Questions

Q. 1. Evaluate : $\Delta = \begin{vmatrix} x & x+1 \\ x-1 & x \end{vmatrix}$

[*USEB*, 2013, *CBSE*, 2014 (*Comptt.*)]

Solution : $\Delta = \begin{vmatrix} x & x+1 \\ x-1 & x \end{vmatrix}$

$= (x)(x - 1) - (x - 1)(x + 1)$
$= x^2 - (x^2 - 1)$
$= 1$

Q. 2. Evaluate : $\begin{vmatrix} 4 & 9 & 7 \\ 3 & 5 & 7 \\ 5 & 4 & 5 \end{vmatrix}$ [*JAC*, 2013]

Solution : Let $\Delta = \begin{vmatrix} 4 & 9 & 7 \\ 3 & 5 & 7 \\ 5 & 4 & 5 \end{vmatrix}$

$= 4(25 - 28) - 9(15 - 35) + 7(12 - 25)$
$= -12 + 180 - 91$
$= 77$

Q. 3. Prove that :

$$\begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix} = abc + 2fgh - af^2 - bg^2 - ch^2.$$

Solution :

$\because$ $\Delta = a\begin{vmatrix} b & f \\ f & c \end{vmatrix} - h\begin{vmatrix} h & f \\ g & c \end{vmatrix} + g\begin{vmatrix} h & b \\ g & f \end{vmatrix}$

$= a(bc - f^2) - h(ch - fg) + g(hf - bg)$
$= abc - af^2 - ch^2 + fgh + fgh - bg^2$
$= abc + 2fgh - af^2 - bg^2 - ch^2.$

Q. 4. Write the value of $\begin{vmatrix} 2 & 7 & 65 \\ 3 & 8 & 75 \\ 5 & 9 & 86 \end{vmatrix}$

[*AI CBSE*, 2014 (Comptt.)]

Solution : Let $\Delta = \begin{vmatrix} 2 & 7 & 65 \\ 3 & 8 & 75 \\ 5 & 9 & 86 \end{vmatrix}$

Applying $C_3 \to C_3 - 9C_2$

$$\Delta = \begin{vmatrix} 2 & 7 & 2 \\ 3 & 8 & 3 \\ 5 & 9 & 5 \end{vmatrix}$$

$\Rightarrow \quad \Delta = 0 \quad$ ($\because$ C_1 and C_3 are identical)

Q. 5. Prove that :

$$\begin{vmatrix} b+c & c+a & a+b \\ q+r & r+p & p+q \\ y+z & z+x & x+y \end{vmatrix} = 2\begin{vmatrix} a & b & c \\ p & q & r \\ x & y & z \end{vmatrix}$$

(*CBSE Delhi I*, 2012; *AI CBSE*, 14)

Solution : Let $\Delta = \begin{vmatrix} b+c & c+a & a+b \\ q+r & r+p & p+q \\ y+z & z+x & x+y \end{vmatrix}$

Applying $C_1 \to C_1 + C_2 + C_3$,

$$\Delta = \begin{vmatrix} 2(b+c+a) & c+a & a+b \\ 2(q+r+p) & r+p & p+q \\ 2(y+z+x) & z+x & x+y \end{vmatrix}$$

$$= 2\begin{vmatrix} b+c+a & c+a & a+b \\ q+r+p & r+p & p+q \\ y+z+x & z+x & x+y \end{vmatrix}$$

(Taking common 2 from C_1)

Applying $C_2 \to C_2 - C_1$ and $C_3 \to C_3 - C_1$

$$= 2\begin{vmatrix} b+c+a & -b & -c \\ q+r+p & -q & -r \\ y+z+x & -y & -z \end{vmatrix}$$

$$= 2(-1)^2\begin{vmatrix} b+c+a & b & c \\ q+r+p & q & r \\ y+z+x & y & z \end{vmatrix}$$

$$= 2\begin{vmatrix} b+c+a & b & c \\ q+r+p & q & r \\ y+z+x & y & z \end{vmatrix}$$

Applying $C_1 \to C_1 - C_2 - C_3$

$$= 2\begin{vmatrix} a & b & c \\ p & q & r \\ x & y & z \end{vmatrix}$$

Q. 6. If $\Delta = \begin{vmatrix} 5 & 3 & 8 \\ 2 & 0 & 1 \\ 1 & 2 & 3 \end{vmatrix}$, write the minor of the element a_{23}. (*CBSE, Delhi*, 2012)

Solution :

Minor of the element a_{23} in Δ

$$= \begin{vmatrix} 5 & 3 \\ 1 & 2 \end{vmatrix}$$

$$= 10 - 3 = 7$$

Q. 7. If $\Delta = \begin{vmatrix} 5 & 3 & 8 \\ 2 & 0 & 1 \\ 1 & 2 & 3 \end{vmatrix}$, write the co-factor of the element a_{32}. (*CBSE, Delhi*, 2012)

Solution :

We have

$$\Delta = \begin{vmatrix} 5 & 3 & 8 \\ 2 & 0 & 1 \\ 1 & 2 & 3 \end{vmatrix}$$

co-factor of the element a_{32} in Δ

$$= (-1)^{3+2}\begin{vmatrix} 5 & 8 \\ 2 & 1 \end{vmatrix}$$

$$= -(5 - 16) = -(-11) = 11.$$

Q. 8. Find the value of x if $\begin{vmatrix} x & 7 \\ x & x \end{vmatrix} = -10$ (*BSEB*, 2014)

Solution : $\begin{vmatrix} x & 7 \\ x & x \end{vmatrix} = -10$

$\Rightarrow \quad (x)(x) - (7)(x) = -10$

$\Rightarrow \quad x^2 - 7x = -10$

$\Rightarrow \quad x^2 - 7x + 10 = 0$

$\Rightarrow \quad (x-2)(x-5) = 0$

$\therefore \quad x = 2, 5$

Q. 9. Let A be a square matrix of order 3 × 3. Write the value of $|2A|$, where $|A| = 4$. (*CBSE, Delhi*, 2012)

Solution :

$\because$ A be a square matrix of order 3 × 3, therefore

$|2A| = 2 \times 2 \times 2\,|A|$

$= 8 \times 4$

$= 32$

Q. 10. Find the value of x if $\begin{vmatrix} 3 & x \\ x & 1 \end{vmatrix} = \begin{vmatrix} 3 & 2 \\ 4 & 1 \end{vmatrix}$ (*USEB*, 2014)

Solution : $\begin{vmatrix} 3 & x \\ x & 1 \end{vmatrix} = \begin{vmatrix} 3 & 2 \\ 4 & 1 \end{vmatrix}$

$\Rightarrow \quad 3 - x^2 = 3 - 8$

$\Rightarrow \quad 3 - x^2 = -5$

$\Rightarrow \quad x^2 = 9$

$\therefore \quad x = \pm 3$

Q. 11. If $\begin{vmatrix} x+1 & x-1 \\ x-3 & x+2 \end{vmatrix} = \begin{vmatrix} 4 & -1 \\ 1 & 3 \end{vmatrix}$, then write the value of x. (*CBSE*, 2013)

Solution : $\begin{vmatrix} x+1 & x-1 \\ x-3 & x+2 \end{vmatrix} = \begin{vmatrix} 4 & -1 \\ 1 & 3 \end{vmatrix}$

$\Rightarrow (x+1)(x+2) - (x-3)(x-1) = 12 + 1$

$\Rightarrow \quad (x^2 + 3x + 2) - (x^2 - 4x + 3) = 13$

$\Rightarrow \quad 7x - 1 = 13$

$\Rightarrow \quad 7x = 14$

$\therefore \quad x = 2$

Q. 12. If $\begin{vmatrix} 3x & 7 \\ -2 & 4 \end{vmatrix} = \begin{vmatrix} 8 & 7 \\ 6 & 4 \end{vmatrix}$**, then find the value of** x**.**

(*AI CBSE*, 2014)

Solution : $\begin{vmatrix} 3x & 7 \\ -2 & 4 \end{vmatrix} = \begin{vmatrix} 8 & 7 \\ 6 & 4 \end{vmatrix}$

$\Rightarrow \quad (3x)(4) - (-2)(7) = 32 - 42$

$\Rightarrow \quad 12x + 14 = -10$

$\Rightarrow \quad 12x = -24$

$\therefore \quad x = -2$

Q. 13. If $\begin{vmatrix} 2x & 5 \\ 8 & x \end{vmatrix} = \begin{vmatrix} 6 & -2 \\ 7 & 3 \end{vmatrix}$**, write the value of** x**.**

(*CBSE*, 2014)

Solution : $\begin{vmatrix} 2x & 5 \\ 8 & x \end{vmatrix} = \begin{vmatrix} 6 & -2 \\ 7 & 3 \end{vmatrix}$

$\Rightarrow \quad 2x^2 - 40 = 18 + 14$

$\Rightarrow \quad 2x^2 = 72$

$\Rightarrow \quad x^2 = 36$

$\therefore \quad x = \pm 6$

Q. 14. Show that $x = 1$ **is a root of the equation**

$$\begin{vmatrix} x+1 & 2x & 11 \\ 2x & x+1 & -4 \\ -3 & 4x-7 & 6 \end{vmatrix} = 0$$

(*JAC*, 2014)

Solution : Putting $x = 1$ in

$$\begin{vmatrix} x+1 & 2x & 11 \\ 2x & x+1 & -4 \\ -3 & 4x-7 & 6 \end{vmatrix} = 0, \text{ we get}$$

$$\begin{vmatrix} 2 & 2 & 11 \\ 2 & 2 & -4 \\ -3 & -3 & 6 \end{vmatrix} = 0$$

$\Rightarrow \quad 0 = 0$

(LHS = 0 as C_1 and C_2 are identical)

which is true

Hence the result.

➠ Short Answer Type Questions

Q. 1. If a, b, c **are in A.P., then find the value of the determinant :**

(*BSEB*, 2013)

$$\begin{vmatrix} x+2 & x+3 & x+2a \\ x+3 & x+4 & x+2b \\ x+4 & x+5 & x+2c \end{vmatrix}$$

Solution : Let $\Delta = \begin{vmatrix} x+2 & x+3 & x+2a \\ x+3 & x+4 & x+2b \\ x+4 & x+5 & x+2c \end{vmatrix}$

Applying $R_1 \to R_1 + R_3 - 2R_2$

$$\Delta = \begin{vmatrix} 0 & 0 & 2a+2c-4b \\ x+3 & x+4 & x+2b \\ x+4 & x+5 & x+2c \end{vmatrix}$$

$$= \begin{vmatrix} 0 & 0 & 2(a+c-2b) \\ x+3 & x+4 & x+2b \\ x+4 & x+5 & x+2c \end{vmatrix}$$

$$= \begin{vmatrix} 0 & 0 & 0 \\ x+3 & x+4 & x+2b \\ x+4 & x+5 & x+2c \end{vmatrix}$$

$\because$ a, b, c are in A.P.

$\therefore \quad 2b = a + c$

$\Rightarrow \quad a + c - 2b = 0$

$= 0 \qquad$ ($\because$ each element of R_1 is 0)

Q. 2. Using property of determinants, show that

$$\begin{vmatrix} x+y & 2x & 2x \\ 2x & x+y & 2x \\ 2x & 2x & x+y \end{vmatrix} = (5x+y)(y-x)^2$$

(*BSEB*, 2014)

Solution : Let $\Delta = \begin{vmatrix} x+y & 2x & 2x \\ 2x & x+y & 2x \\ 2x & 2x & x+y \end{vmatrix}$

Applying $R_1 \to R_1 + R_2 + R_3$

$$\Delta = \begin{vmatrix} 5x+y & y & 5x+y \\ 2x & x+y & 2x \\ 2x & 2x & x+y \end{vmatrix}$$

$$= (5x+y)\begin{vmatrix} 1 & 1 & 1 \\ 2x & x+y & 2x \\ 2x & 2x & x+y \end{vmatrix}$$

[Taking out common $(5x+y)$ from R_1]

Applying $C_2 \to C_2 - C_1$ and $C_3 \to C_3 - C_1$

$$\Delta = (5x+y)\begin{vmatrix} 1 & 0 & 0 \\ 2x & y-x & 0 \\ 3x & 0 & y-x \end{vmatrix}$$

$$= (5x+y)\begin{vmatrix} y-x & 0 \\ 0 & y-x \end{vmatrix}$$

(expanding along R_1)

$= (5x+y)(y-x)^2$

Q. 3. Express the determinant $|A|$ **in factors, where :**

$$|A| = \begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ a^3 & b^3 & c^3 \end{vmatrix} = (a-b)(b-c)(c-a)(a+b+c).$$

(*CBSE, Delhi II*, 2012)

Solution : We have $|A| = \begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ a^3 & b^3 & c^3 \end{vmatrix}$

Applying $C_1 \to C_1 - C_2$ and $C_2 \to C_2 - C_3$

$$|A| = \begin{vmatrix} 0 & 0 & 1 \\ a-b & b-c & c \\ a^3-b^3 & b^3-c^3 & c^3 \end{vmatrix}$$

$$= \begin{vmatrix} 0 & 0 & 1 \\ a-b & b-c & c \\ (a-b)(a^2+ab+b^2) & (b-c)(b^2+bc+c^2) & c^3 \end{vmatrix}$$

$$= (a-b)(b-c)\begin{vmatrix} 0 & 0 & 1 \\ 1 & 1 & c \\ a^2+ab+b^2 & b^2+bc+c^2 & c^3 \end{vmatrix}$$

Expanding along R_1,

$= (a-b)(b-c).1\ [1(b^2+bc+c^2) - 1(a^2+ab+b^2)]$

$= (a-b)(b-c) \quad [b(c-a)+(c-a)(c+a)]$

$= (a-b)(b-c)(c-a)(a+b+c).$

Q. 4. Show that :

$$\begin{vmatrix} 1+a^2-b^2 & 2ab & -2b \\ 2ab & 1-a^2+b^2 & 2a \\ 2b & -2a & 1-a^2-b^2 \end{vmatrix} = (1+a^2+b^2)^3$$

(*CBSE, Delhi*, 2009, *BSER*, 2013)

Solution :

$$\text{Let } \Delta = \begin{vmatrix} 1+a^2-b^2 & 2ab & -2b \\ 2ab & 1-a^2+b^2 & 2a \\ 2b & -2a & 1-a^2-b^2 \end{vmatrix}$$

$$\text{then, } \Delta = \begin{vmatrix} 1+a^2+b^2 & 0 & -2b \\ 0 & 1+a^2+b^2 & 2a \\ b(1+a^2+b^2) & -a(1+a^2+b^2) & 1-a^2-b^2 \end{vmatrix},$$

(applying $C_1 \to C_1 - bC_3$, $C_2 \to C_2 + aC_3$)

$$= (1+a^2+b^2)^2 \begin{vmatrix} 1 & 0 & -2b \\ 0 & 1 & 2a \\ b & -a & 1-a^2-b^2 \end{vmatrix}$$

(Taking out common $1 + a^2 + b^2$ from C_1, C_2)

Expanding along C_1,

$\Delta = (1+a^2+b^2)^2\ [1.(1-a^2-b^2+2a^2) + b\,(0+2b)]$

$= (1+a^2+b^2)^2\,(1+a^2+b^2) = (1+a^2+b^2)^3.$

Q. 5. Show that $\Delta = \Delta_1$ where

$$\Delta = \begin{vmatrix} Ax & x^2 & 1 \\ By & y^2 & 1 \\ Cz & z^2 & 1 \end{vmatrix}, \Delta_1 = \begin{vmatrix} A & B & C \\ x & y & z \\ zy & zx & xy \end{vmatrix}$$

[*AI CBSE*, 2014 (*Comptt.*)]

Solution :

We have

$$\Delta_1 = \begin{vmatrix} A & B & C \\ x & y & z \\ zy & zx & xy \end{vmatrix}$$

Operating $C_1 \to xC_1$, $C_2 \to yC_2$, $C_3 \to zC_3$ and therefore dividing the determinant by xyz,

$$\Delta_1 = \frac{1}{xyz}\begin{vmatrix} Ax & By & Cz \\ x^2 & y^2 & z^2 \\ xyz & xyz & xyz \end{vmatrix}$$

$$= \frac{xyz}{xyz}\begin{vmatrix} Ax & By & Cz \\ x^2 & y^2 & z^2 \\ 1 & 1 & 1 \end{vmatrix}$$

(Taking out common xyz from C_3)

$$= \begin{vmatrix} Ax & By & Cz \\ x^2 & y^2 & z^2 \\ 1 & 1 & 1 \end{vmatrix}$$

$$= \begin{vmatrix} Ax & x^2 & 1 \\ By & y^2 & 1 \\ Cz & z^2 & 1 \end{vmatrix}$$

(by property 1)

Q. 6. Using properties of determinants, prove that :

$$\begin{vmatrix} a & b & c \\ a^2 & b^2 & c^2 \\ bc & ca & ab \end{vmatrix} = (a-b)(b-c)(c-a)(ab+bc+ca)$$

[*CBSE*, 2013 (*Comptt.*)]

Solution :

$$\text{Let } \Delta = \begin{vmatrix} a & b & c \\ a^2 & b^2 & c^2 \\ bc & ca & ab \end{vmatrix}$$

Applying $C_1 \to C_1 - C_2$ and $C_2 \to C_2 - C_3$

$$\Delta = \begin{vmatrix} a-b & b-c & c \\ (a-b)(a+b) & (b-c)(b+c) & c^2 \\ -c(a-b) & -a(b-c) & ab \end{vmatrix}$$

$$= (a-b)(b-c)\begin{vmatrix} 1 & 1 & c \\ a+b & b+c & c^2 \\ -c & -a & ab \end{vmatrix}$$

[Taking out common $a-b$, $b-c$ from C_1, C_2 respectively]

Applying $C_1 \to C_1 - C_2$

$$= (a-b)(b-c)\begin{vmatrix} 0 & 1 & c \\ a-c & b+c & c^2 \\ a-c & -a & ab \end{vmatrix}$$

$$= (a-b)(b-c)(a-c)\begin{vmatrix} 0 & 1 & c \\ 1 & b+c & c^2 \\ 1 & -a & ab \end{vmatrix}$$

(Taking out common $(a-c)$ from C_1)

Applying $R_2 \to R_2 - R_3$

$$= (a-b)(b-c)(a-c)\begin{vmatrix} 0 & 1 & c \\ 0 & a+b+c & c^2-ab \\ 1 & -a & ab \end{vmatrix}$$

$$= (a-b)(b-c)(a-c)\begin{vmatrix} 1 & c \\ a+b+c & c^2-ab \end{vmatrix}$$

Expanding along C_1.

$= (a-b)(b-c)(a-c)\,\{(c^2-ab) - c(a+b+c)\}$

$= (a-b)(b-c)(a-c)(-ab-ca-cb)$

$= (a-b)(b-c)(c-a)(ab+bc+ca)$

Q. 7. Without expanding the determinant show that :

$$\begin{vmatrix} x+y & x & x \\ 5x+4y & 4x & 2x \\ 10x+8y & 8x & 3x \end{vmatrix} = x^3$$

(*CBSE, AI*, 2009, 14)

Solution :

Let $\Delta = \begin{vmatrix} x+y & x & x \\ 5x+4y & 4x & 2x \\ 10x+8y & 8x & 3x \end{vmatrix}$

then $\Delta = \begin{vmatrix} x & x & x \\ 5x & 4x & 2x \\ 10x & 8x & 3x \end{vmatrix} + \begin{vmatrix} y & x & x \\ 4y & 4x & 2x \\ 8y & 8x & 3x \end{vmatrix}$ (from property 4)

$= \Delta_1 + \Delta_2$

where $\Delta_1 = \begin{vmatrix} x & x & x \\ 5x & 4x & 2x \\ 10x & 8x & 3x \end{vmatrix}$ and $\Delta_2 = \begin{vmatrix} y & x & x \\ 4y & 4x & 2x \\ 8y & 8x & 3x \end{vmatrix}$

Now, $\Delta_1 = \begin{vmatrix} x & x & x \\ 5x & 4x & 2x \\ 10x & 8x & 3x \end{vmatrix}$

$= x^3 \begin{vmatrix} 1 & 1 & 1 \\ 5 & 4 & 2 \\ 10 & 8 & 3 \end{vmatrix}$

(Taking out common x from C_1, C_2, C_3)

Applying $C_2 \to C_2 - C_1$ and $C_3 \to C_3 - C_1$

$= x^3 \begin{vmatrix} 1 & 0 & 0 \\ 5 & -1 & -3 \\ 10 & -2 & -7 \end{vmatrix}$

$= x^3 \begin{vmatrix} -1 & -3 \\ -2 & -7 \end{vmatrix}$ (Expanding along R_1)

$= x^3 (7 - 6) = x^3$

$\Delta_2 = \begin{vmatrix} y & x & x \\ 4y & 4x & 2x \\ 8y & 8x & 3x \end{vmatrix}$

$= x^2 y \begin{vmatrix} 1 & 1 & 1 \\ 4 & 4 & 2 \\ 8 & 8 & 3 \end{vmatrix}$

(Taking out common y, x, x from C_1, C_2, C_3 respectively)

$= x^2y\,(0)$ ($\because$ C_1 and C_2 are identical)

$= 0$

$\therefore$ $\Delta = \Delta_1 + \Delta_2$

$\Rightarrow$ $\Delta = x^3 + 0$

$\Rightarrow$ $\Delta = x^3$

Q. 8. Factorize : $\begin{vmatrix} a-b-c & 2a & 2a \\ 2b & b-c-a & 2b \\ 2c & 2c & c-a-b \end{vmatrix}$

(*JAC*, 2013)

Solution :

Let $\Delta = \begin{vmatrix} a-b-c & 2a & 2a \\ 2b & b-c-a & 2b \\ 2c & 2c & c-a-b \end{vmatrix}$

Applying $R_1 \to R_1 + R_2 + R_3$

$\Delta = \begin{vmatrix} a+b+c & a+b+c & a+b+c \\ 2b & b-c-a & 2b \\ 2c & 2c & c-a-b \end{vmatrix}$

$\Rightarrow$ $\Delta = (a+b+c) \begin{vmatrix} 1 & 1 & 1 \\ 2b & b-c-a & 2b \\ 2c & 2c & c-a-b \end{vmatrix}$

[Taking out common $(a + b + c)$ from R_1]

Applying $C_2 \to C_2 - C_1$ and $C_3 \to C_3 - C_1$

$\Delta = (a+b+c) \begin{vmatrix} 1 & 0 & 0 \\ 2b & -(a+b+c) & 0 \\ 2c & 0 & -(a+b+c) \end{vmatrix}$

$= (a+b+c) \begin{vmatrix} -(a+b+c) & 0 \\ 0 & -(a+b+c) \end{vmatrix}$

Expanding along R_1

$= (a+b+c)^3$

Q. 9. Using properties of determinants, prove the following : (*CBSE*, 2013; 14 (*Comptt.*))

$$\begin{vmatrix} 1 & x & x^2 \\ x^2 & 1 & x \\ x & x^2 & 1 \end{vmatrix} = (1-x^3)^2$$

Solution : Let $\Delta = \begin{vmatrix} 1 & x & x^2 \\ x^2 & 1 & x \\ x & x^2 & 1 \end{vmatrix}$

Applying $R_1 \to R_1 - R_2$ and $R_3 \to R_3 - R_2$

$\Delta = \begin{vmatrix} 1-x^2 & x-1 & x^2-x \\ x^2 & 1 & x \\ x-x^2 & x^2-1 & 1-x \end{vmatrix}$

$= \begin{vmatrix} (1-x)(1+x) & -(1-x) & -x(1-x) \\ x^2 & 1 & x \\ x(1-x) & -(1-x)(1+x) & 1-x \end{vmatrix}$

$= (1-x)^2 \begin{vmatrix} 1+x & -1 & -x \\ x^2 & 1 & x \\ x & -(1+x) & 1 \end{vmatrix}$

[Taking out common $(1 - x)$ from R_1 and R_3]

Applying $C_1 \to C_1 + C_2 + C_3$

$= (1-x)^2 \begin{vmatrix} 0 & -1 & -x \\ 1+x+x^2 & 1 & x \\ 0 & -(1+x) & 1 \end{vmatrix}$

$= (1-x)^2 \left\{ -(1+x+x^2) \begin{vmatrix} -1 & -x \\ -(1+x) & 1 \end{vmatrix} \right\}$

$= -(1-x)^2 (1+x+x^2) \{-(1+x+x^2)\}$

$= (1-x)^2 (1+x+x^2)^2$

$= \{(1-x)(1+x+x^2)\}^2$

$= (1-x^3)^2$

Q. 10. Using properties of determinants, show that :

$$\begin{vmatrix} b+c & a & a \\ b & c+a & b \\ c & c & a+b \end{vmatrix} = 4abc.$$

[*CBSE, Delhi*, 2012; *BSEB, USEB*, 2013; *AI CBSE*, 2014 (*Comptt.*)]

Solution :

$$\text{LHS} = \begin{vmatrix} b+c & a & a \\ b & c+a & b \\ c & c & a+b \end{vmatrix}$$

Applying $R_1 \to R_1 + R_2 + R_3$

$$= \begin{vmatrix} 2(b+c) & 2(c+a) & 2(a+b) \\ b & c+a & b \\ c & c & a+b \end{vmatrix}$$

Taking 2 as common from R_1

$$= 2\begin{vmatrix} b+c & c+a & a+b \\ b & c+a & b \\ c & c & a+b \end{vmatrix},$$

Applying $R_2 \to R_2 - R_1$ and $R_3 \to R_3 - R_1$

$$= 2\begin{vmatrix} b+c & c+a & a+b \\ -c & 0 & -a \\ -b & -a & 0 \end{vmatrix}$$

Applying $R_1 \to R_1 + R_2 + R_3$

$$= 2\begin{vmatrix} 0 & c & b \\ -c & 0 & -a \\ -b & -a & 0 \end{vmatrix}$$

$= 2\,[0 - c\,(0 - ab) + b\,(ac - 0)]$

$= 2(abc + abc)$

$= 2 \times 2abc = 4abc.$

Hence proved.

Long Answer Type Questions

Q. 1. Using properties of determinants, prove the following : (*AI CBSE*, 2013)

$$\begin{vmatrix} x & x+y & x+2y \\ x+2y & x & x+y \\ x+y & x+2y & x \end{vmatrix} = 9y^2\,(x+y)$$

Solution :

Let $$\Delta = \begin{vmatrix} x & x+y & x+2y \\ x+2y & x & x+y \\ x+y & x+2y & x \end{vmatrix}$$

Applying $C_1 \to C_1 + C_2 + C_3$

$$\Delta = \begin{vmatrix} 3(x+y) & x+y & x+2y \\ 3(x+y) & x & x+y \\ 3(x+y) & x+2y & x \end{vmatrix}$$

$$= 3\,(x+y)\begin{vmatrix} 1 & x+y & x+2y \\ 1 & x & x+y \\ 1 & x+2y & x \end{vmatrix}$$

[Taking out common $3\,(x+y)$ from C_1]

Applying $R_2 \to R_2 - R_1$ and $R_3 \to R_3 - R_1$,

$$\Delta = 3\,(x+y)\begin{vmatrix} 1 & x+y & x+2y \\ 0 & -y & -y \\ 0 & y & -2y \end{vmatrix}$$

$$= 3\,(x+y)\begin{vmatrix} -y & -y \\ y & -2y \end{vmatrix}$$

(Expanding along C_1)

$= 3\,(x+y)\,(2y^2 + y^2)$

$= 9y^2\,(x+y)$

Q. 2. Using properties of determinants, prove the following : (*AI CBSE*, 2013)

$$\begin{vmatrix} 3x & -x+y & -x+z \\ x-y & 3y & z-y \\ x-z & y-z & 3z \end{vmatrix} = 3\,(x+y+z)\,(xy+yz+zx)$$

Solution :

Let $$\Delta = \begin{vmatrix} 3x & -x+y & -x+z \\ x-y & 3y & z-y \\ x-z & y-z & 3z \end{vmatrix}$$

Applying $C_1 \to C_1 + C_2 + C_3$

$$\Delta = \begin{vmatrix} x+y+z & -x+y & -x+z \\ x+y+z & 3y & z-y \\ x+y+z & y-z & 3z \end{vmatrix}$$

$$= (x+y+z)\begin{vmatrix} 1 & -x+y & -x+z \\ 1 & 3y & z-y \\ 1 & y-z & 3z \end{vmatrix}$$

[Taking out common $(x+y+z)$ from C_1]

Applying $R_2 \to R_2 - R_1$ and $R_3 \to R_3 - R_1$

$$\Delta = (x+y+z)\begin{vmatrix} 1 & -x+y & -x+z \\ 0 & 2y+x & -y+x \\ 0 & -z+x & 2z+x \end{vmatrix}$$

$$= (x+y+z)\begin{vmatrix} 2y+x & -y+x \\ -z+x & 2z+x \end{vmatrix}$$

(Expanding along C_1)

Applying $C_1 \to C_1 - C_2$

$$= (x+y+z)\begin{vmatrix} 3y & -y+x \\ -3z & 2z+x \end{vmatrix}$$

$$= 3\,(x+y+z)\begin{vmatrix} y & -y+x \\ -z & 2z+x \end{vmatrix}$$

(Taking out common 3 from C_1)

$= 3\,(x+y+z)\,\{y\,(2z+x) + z\,(-y+x)\}$

$= 3\,(x+y+z)\,(2yz + xy - yz + zx)$

$= 3\,(x+y+z)\,(xy + yz + zx)$

Q. 3. If x, y, z are different and

$$\Delta = \begin{vmatrix} x & x^2 & 1+x^3 \\ y & y^2 & 1+y^3 \\ z & z^2 & 1+z^3 \end{vmatrix} = 0, \text{ then show that } 1 + xyz = 0.$$

(*CBSE, AI, Compartment*, 2009; *USEB*, 2009, 14; *CBSE, Delhi*, 2008)

Solution :

Given determinant can be written as,

$$\Delta = \begin{vmatrix} x & x^2 & 1 \\ y & y^2 & 1 \\ z & z^2 & 1 \end{vmatrix} + \begin{vmatrix} x & x^2 & x^3 \\ y & y^2 & y^3 \\ z & z^2 & x^3 \end{vmatrix}$$

$$= \begin{vmatrix} x & x^2 & 1 \\ y & y^2 & 1 \\ z & z^2 & 1 \end{vmatrix} + xyz \begin{vmatrix} 1 & x & x^2 \\ 1 & y & y^2 \\ 1 & z & z^2 \end{vmatrix}$$

$$= (-1)^2 \begin{vmatrix} 1 & x & x^2 \\ 1 & y & y^2 \\ 1 & z & z^2 \end{vmatrix} + xyz \begin{vmatrix} 1 & x & x^2 \\ 1 & y & y^2 \\ 1 & z & z^2 \end{vmatrix}$$

$$= \begin{vmatrix} 1 & x & x^2 \\ 1 & y & y^2 \\ 1 & z & z^2 \end{vmatrix} (1 + xyz)$$

Now $\begin{vmatrix} 1 & x & x^2 \\ 1 & y & y^2 \\ 1 & z & z^2 \end{vmatrix} = \begin{vmatrix} 1 & x & x^2 \\ 0 & y-x & y^2-x^2 \\ 0 & z-x & z^2-x^2 \end{vmatrix}$

(applying $R_2 \to R_2 - R_1$ and $R_3 \to R_3 - R_1$)

$$= (y-x)(z-x) \begin{vmatrix} 1 & x & x^2 \\ 0 & 1 & y+x \\ 0 & 1 & z+x \end{vmatrix}$$

$$= (y-x)(z-x).1\{(z+x) - 1(y+x)\}$$

(Expanding along C_1)

$$= (y-x)(z-x)(z-y) = (x-y)(y-z)(z-x)$$

$\therefore \quad \Delta = (1 + xyz)(x-y)(y-z)(z-x)$

Given $\quad \Delta = 0,$

$\therefore (1 + xyz)(x-y)(y-z)(z-x) = 0$

But $\quad x \neq y \neq z \Rightarrow (x-y)(y-z)(z-x) \neq 0,$

$\therefore \quad 1 + xyz = 0$

Q. 4. Using properties of determinants, prove the following : (*CBSE*, 2014)

$$\begin{vmatrix} x^2+1 & xy & xz \\ xy & y^2+1 & yz \\ xz & yz & z^2+1 \end{vmatrix} = 1 + x^2 + y^2 + z^2$$

Solution : Let $\Delta = \begin{vmatrix} x^2+1 & xy & xz \\ xy & y^2+1 & yz \\ xz & yz & z^2+1 \end{vmatrix}$

Multiplying R_1, R_2, R_3 by x, y, z repectively and therefore dividing the determinant by xyz,

$$\Delta = \frac{1}{xyz} \begin{vmatrix} x(x^2+1) & x^2y & x^2z \\ xy^2 & y(y^2+1) & y^2z \\ xz^2 & yz^2 & z(z^2+1) \end{vmatrix}$$

(Taking out common x, y, z from C_1, C_2, C_3 respectively)

$$\Delta = \frac{xyz}{xyz} \begin{vmatrix} x^2+1 & x^2 & x^2 \\ y^2 & y^2+1 & y^2 \\ z^2 & z^2 & z^2+1 \end{vmatrix}$$

Applying $R_1 \to R_1 + R_2 + R_3$,

$$\Delta = \begin{vmatrix} 1+x^2+y^2+z^2 & 1+x^2+y^2+z^2 & 1+x^2+y^2+z^2 \\ y^2 & y^2+1 & y^2 \\ z^2 & z^2 & z^2+1 \end{vmatrix}$$

$$= (1 + x^2 + y^2 + z^2) \begin{vmatrix} 1 & 1 & 1 \\ y^2 & y^2+1 & y^2 \\ z^2 & z^2 & z^2+1 \end{vmatrix}$$

[Taking out $(1+x^2+y^2+z^2)$ common from R_1]

Applying $C_2 \to C_2 - C_1$ and $C_3 \to C_3 - C_1$

$$\Delta = (1 + x^2 + y^2 + z^2) \begin{vmatrix} 1 & 0 & 0 \\ y^2 & 1 & 0 \\ z^2 & 0 & 1 \end{vmatrix}$$

$$= (1 + x^2 + y^2 + z^2) \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} \quad \text{(Expanding along } R_1\text{)}$$

$$= (1 + x^2 + y^2 + z^2)(1 - 0)$$

$$= 1 + x^2 + y^2 + z^2$$

Q. 5. Prove the following using properties of determinants :

$$\begin{vmatrix} a+b+2c & a & b \\ c & b+c+2a & b \\ c & a & c+a+2b \end{vmatrix} = 2(a+b+c)^3$$

(*CBSE*, 2012, 14)

Solution: Let $\Delta = \begin{vmatrix} a+b+2c & a & b \\ c & b+c+2a & b \\ c & a & c+a+2b \end{vmatrix}$

Applying $C_1 \to C_1 + C_2 + C_3$

$$\Delta = \begin{vmatrix} 2(a+b+c) & a & b \\ 2(a+b+c) & b+c+2a & b \\ 2(a+b+c) & a & c+a+2b \end{vmatrix}$$

$$= 2(a+b+c) \begin{vmatrix} 1 & a & b \\ 1 & b+c+2a & b \\ 1 & a & c+a+2b \end{vmatrix}$$

[Taking out common $2(a+b+c)$ from C_1]

Applying $R_2 \to R_2 - R_1$ and $R_3 \to R_3 - R_1$

$$= 2(a+b+c) \begin{vmatrix} 1 & a & b \\ 0 & a+b+c & 0 \\ 0 & 0 & a+b+c \end{vmatrix}$$

$$= 2(a+b+c) \begin{vmatrix} a+b+c & 0 \\ 0 & a+b+c \end{vmatrix}$$

(Expanding along C_1)

$$= 2(a+b+c)(a+b+c)^2$$

$$= 2(a+b+c)^3$$

Q. 6. Prove that :

$$\begin{vmatrix} 1+a & 1 & 1 \\ 1 & 1+b & 1 \\ 1 & 1 & 1+c \end{vmatrix} = abc\left(1+\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)$$

$$= abc + bc + ca + ab$$

(*BSER, AI CBSE*, 2014)

Solution : Let $\Delta = \begin{vmatrix} 1+a & 1 & 1 \\ 1 & 1+b & 1 \\ 1 & 1 & 1+c \end{vmatrix}$

(Taking a, b, c common from R_1, R_2, R_3 respectively)

$$\Delta = abc \begin{vmatrix} 1+\frac{1}{a} & \frac{1}{a} & \frac{1}{a} \\ \frac{1}{b} & 1+\frac{1}{b} & \frac{1}{b} \\ \frac{1}{c} & \frac{1}{c} & 1+\frac{1}{c} \end{vmatrix}$$

Applying $R_1 \to R_1 + R_2 + R_3$

$$\Delta = abc \begin{vmatrix} 1+\frac{1}{a}+\frac{1}{b}+\frac{1}{c} & 1+\frac{1}{a}+\frac{1}{b}+\frac{1}{c} & 1+\frac{1}{a}+\frac{1}{b}+\frac{1}{c} \\ \frac{1}{b} & 1+\frac{1}{b} & \frac{1}{b} \\ \frac{1}{c} & \frac{1}{c} & 1+\frac{1}{c} \end{vmatrix}$$

$$= abc\left(1+\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right) \begin{vmatrix} 1 & 1 & 1 \\ \frac{1}{b} & 1+\frac{1}{b} & \frac{1}{b} \\ \frac{1}{c} & \frac{1}{c} & 1+\frac{1}{c} \end{vmatrix}$$

[Taking out common $\left(1+\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)$ from R_1]

Applying $C_2 \to C_2 - C_1$ and $C_3 \to C_3 - C_1$,

$$\Delta = abc\left(1+\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right) \begin{vmatrix} 1 & 0 & 0 \\ \frac{1}{b} & 1 & 0 \\ \frac{1}{c} & 0 & 1 \end{vmatrix}$$

$$= abc\left(1+\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right) \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix}$$

(Expanding along R_1)

$$= abc\left(1+\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)$$

Q. 7. Solve by matrix method the following system of equations : (*USEB*, 2014)

$$2x - 3y + 5z = 11$$
$$3x + 2y - 4z = -5$$
$$x + y - 2z = -3$$

Solution :

The given system of equations is

$$2x - 3y + 5z = 11$$
$$3x + 2y - 4z = -5$$
$$x + y - 2z = -3$$

$$\Rightarrow \begin{bmatrix} 2 & -3 & 5 \\ 3 & 2 & -4 \\ 1 & 1 & -2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 11 \\ -5 \\ -3 \end{bmatrix}$$

$\Rightarrow$ $AX = B$, where

$$A = \begin{bmatrix} 2 & -3 & 5 \\ 3 & 2 & -4 \\ 1 & 1 & -2 \end{bmatrix}$$

$$X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

and $$B = \begin{bmatrix} 11 \\ -5 \\ -3 \end{bmatrix}$$

$$|A| = \begin{vmatrix} 2 & -3 & 5 \\ 3 & 2 & -4 \\ 1 & 1 & -2 \end{vmatrix}$$

$$= 2(-4+4) + 3(-6+4) + 5(3-2)$$
$$= 0 - 6 + 5 = -1 \neq 0$$

$\Rightarrow$ A is invertible.

$\Rightarrow$ A^{-1} exists.

Now, $$A^{-1} = \frac{\text{adj.} A}{|A|}$$

To find adj. A :

$$C_{11} = \begin{vmatrix} 2 & -4 \\ 1 & -2 \end{vmatrix} = 0$$

$$C_{12} = -\begin{vmatrix} 3 & -4 \\ 1 & -2 \end{vmatrix} = 2$$

$$C_{13} = \begin{vmatrix} 3 & 2 \\ 1 & 1 \end{vmatrix} = 1$$

$$C_{21} = \begin{vmatrix} -3 & 5 \\ 1 & -2 \end{vmatrix} = -1$$

$$C_{22} = \begin{vmatrix} 2 & 5 \\ 1 & -2 \end{vmatrix} = -9$$

$$C_{23} = -\begin{vmatrix} 2 & -3 \\ 1 & 1 \end{vmatrix} = -5$$

$$C_{31} = \begin{vmatrix} -3 & 5 \\ 2 & -4 \end{vmatrix} = 2$$

$$C_{32} = -\begin{vmatrix} 2 & 5 \\ 3 & -4 \end{vmatrix} = 23$$

$$C_{33} = \begin{vmatrix} 2 & -3 \\ 3 & 2 \end{vmatrix} = 13$$

$$\therefore \quad \text{adj} A = \begin{bmatrix} 0 & 2 & 1 \\ -1 & -9 & -5 \\ 2 & 23 & 13 \end{bmatrix}'$$

$$= \begin{bmatrix} 0 & -1 & 2 \\ 2 & -9 & 23 \\ 1 & -5 & 13 \end{bmatrix}$$

$$\therefore \quad A^{-1} = -\begin{bmatrix} 0 & -1 & 2 \\ 2 & -9 & 23 \\ 1 & -5 & 13 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 1 & -2 \\ -2 & 9 & -23 \\ -1 & 5 & -13 \end{bmatrix}$$

The solution is given by

$$X = A^{-1} B$$

$$\Rightarrow \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 & 1 & -2 \\ -2 & 9 & -23 \\ -1 & 5 & -13 \end{bmatrix} \begin{bmatrix} 11 \\ -5 \\ -3 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0-5+6 \\ -22-45+69 \\ -11-25+39 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

$$\Rightarrow x = 1, y = 2, z = 3$$

(by determination of equality of two matrices)

Q. 8. Using Cramer's rule, solve the following system of linear equations : (*CBSE, Delhi*, 2012)

$$\mathbf{x - y + 2z = 7}$$
$$\mathbf{3x + 4y - 5z = -5}$$
$$\mathbf{2x - y + 3z = 12}$$

Solution : Let $D = \begin{vmatrix} 1 & -1 & 2 \\ 3 & 4 & -5 \\ 2 & -1 & 3 \end{vmatrix}$

$= 1(12 - 5) + 1(9 + 10) + 2(-3 - 8)$

$= 7 + 19 - 22$

$= 26 - 22$

$= 4 \neq 0.$

$\because \quad D \neq 0$

$\therefore$ Solution exists.

Now $D_x = \begin{vmatrix} 7 & -1 & 2 \\ -5 & 4 & -5 \\ 12 & -1 & 3 \end{vmatrix}$

$= 7(12 - 5) + 1(-15 + 60) + 2(5 - 48)$

$= 7 \times 7 + 45 + 2 \times (-43)$

$= 49 + 45 - 86$

$= 94 - 86 = 8$

$D_y = \begin{vmatrix} 1 & 7 & 2 \\ 3 & -5 & -5 \\ 2 & 12 & 3 \end{vmatrix}$

$= 1(-15 + 60) - 7(9 + 10) + 2(36 + 10)$

$= 45 - 7 \times 19 + 2 \times 46$

$= 45 - 133 + 92$

$= 137 - 133 = 4$

$Dz = \begin{vmatrix} 1 & -1 & 7 \\ 3 & 4 & -5 \\ 2 & -1 & 12 \end{vmatrix}$

$= 1(48 - 5) + 1(36 + 10) + 7(-3 - 8)$

$= 43 + 46 - 77$

$= 89 - 77 = 12$

$\therefore \quad x = \dfrac{D_x}{D} = \dfrac{8}{4} = 2$

$y = \dfrac{D_y}{D} = \dfrac{4}{4} = 1$

and $z = \dfrac{D_z}{D} = \dfrac{12}{4} = 3$

Hence, $x = 2, y = 1$ and $z = 3$

Q. 9. Using Cramer's rule, solve the following system of equations :

$$\mathbf{2x + 3y + 3z = 5}$$
$$\mathbf{x - 2y + z = -4}$$
$$\mathbf{3x - y - 2z = 3}$$

(*CBSE, Outside Delhi*, 2012)

Solution :

Let $D = \begin{vmatrix} 2 & 3 & 3 \\ 1 & -2 & 1 \\ 3 & -1 & -2 \end{vmatrix}$

$= 2(4 + 1) - 3(-2 - 3) + 3(-1 + 6)$

$= 2 \times 5 - 3 \times (-5) + 3 \times 5$

$= 10 + 15 + 15 = 40 \neq 0$

$\because \quad D \neq 0$

$\therefore$ Solution exists.

Now $D_x = \begin{vmatrix} 5 & 3 & 3 \\ -4 & -2 & 1 \\ 3 & -1 & -2 \end{vmatrix}$

$= 5(4 + 1) - 3(8 - 3) + 3(4 + 6)$

$= 5 \times 5 - 3 \times 5 + 3 \times 10$

$= 25 - 15 + 30 = 40$

$D_y = \begin{vmatrix} 2 & 5 & 3 \\ 1 & -4 & 1 \\ 3 & 3 & -2 \end{vmatrix}$

$= 2(8 - 3) - 5(-2 - 3) + 3(3 + 12)$

$= 2 \times 5 - 5 \times (-5) + 3 \times 15$

$= 10 + 25 + 45 = 80$

$D_z = \begin{vmatrix} 2 & 3 & 5 \\ 1 & -2 & -4 \\ 3 & -1 & 3 \end{vmatrix}$

$= 2(-6 - 4) - 3(3 + 12) + 5(-1 + 6)$

$= 2(-10) - 3 \times 15 + 5 \times 5$

$= -20 - 45 + 25$

$= -65 + 25 = -40$

Therefore, $x = \dfrac{D_x}{D} = \dfrac{40}{40} = 1$

$y = \dfrac{D_y}{D} = \dfrac{80}{40} = 2$

and $z = \dfrac{D_z}{D} = \dfrac{-40}{40} = 1$

Hence, $x = 1, y = 2$ and $z = -1$

Q. 10. Using Cramer's rule, solve the following system of equations :

$$\mathbf{3x + 4y + 7z = 4}$$
$$\mathbf{2x - y + 3z = -3}$$
$$\mathbf{x + 2y - 3z = 8}$$

(*CBSE, Delhi* III, 2012)

Solution :

Let $D = \begin{vmatrix} 3 & 4 & 7 \\ 2 & -1 & 3 \\ 1 & 2 & -3 \end{vmatrix}$

$= 3(3-6) - 4(-6-3) + 7(4+1)$
$= 3(-3) - 4(-9) + 7 \times 5$
$= -9 + 36 + 35$
$= 62 \neq 0$

$\because \quad D \neq 0$

$\therefore$ Solution exists.

$D_x = \begin{vmatrix} 4 & 4 & 7 \\ -3 & -1 & 3 \\ 8 & 2 & -3 \end{vmatrix}$

$= 4(3-6) - 4(9-24) + 7(-6+8)$
$= 4(-3) - 4(-15) + 7(2)$
$= -12 + 60 + 14$
$= 62$

$D_y = \begin{vmatrix} 3 & 4 & 7 \\ 2 & -3 & 3 \\ 1 & 8 & -3 \end{vmatrix}$

$= 3(9-24) - 4(-6-3) + 7(16+3)$
$= 3(-15) - 4(-9) + 7(19)$
$= -45 + 36 + 133$
$= 124$

and $D_z = \begin{vmatrix} 3 & 4 & 4 \\ 2 & -1 & -3 \\ 1 & 2 & 8 \end{vmatrix}$

$= 3(-8+6) - 4(16+3) + 4(4+1)$
$= 3(-2) - 4(19) + 4(5)$
$= -6 - 76 + 20 = -62$

Therefore, $x = \dfrac{D_x}{D} = \dfrac{62}{62} = 1$

$y = \dfrac{D_y}{D} = \dfrac{124}{62} = 2$

and $z = \dfrac{D_z}{D} = \dfrac{-62}{62} = 1$

Hence, $x = 1, y = 2$ and $z = -1$

Q. 11. Evaluate $A = \begin{vmatrix} 1 & bc & a(b+c) \\ 1 & ca & b(c+a) \\ 1 & ab & c(a+b) \end{vmatrix}$ (*BSEB*, 2015)

Solution : $\because$ $A = \begin{vmatrix} 1 & bc & a(b+c) \\ 1 & ca & b(c+a) \\ 1 & ab & c(a+b) \end{vmatrix}$

Applying $C_3 \to C_2 + C_3$

$A = \begin{vmatrix} 1 & bc & ab+ac+bc \\ 1 & ca & ab+ac+bc \\ 1 & ab & ab+ac+bc \end{vmatrix}$

Taking common $(ab + ac + bc)$

$A = (ab+ac+bc) \begin{vmatrix} 1 & bc & 1 \\ 1 & ca & 1 \\ 1 & ab & 1 \end{vmatrix}$

Since, l_1 and l_3 are identical, thus

$A = (ab + ac + bc) . 0$

$\Rightarrow \quad A = 0$

Q. 12. Without expanding, prove that :

$\begin{vmatrix} x+y & y+z & z+x \\ z & x & y \\ 1 & 1 & 1 \end{vmatrix} = 0$ (*JAC*, 2015)

Solution : Let $A = \begin{vmatrix} x+y & y+z & z+x \\ z & x & y \\ 1 & 1 & 1 \end{vmatrix}$

Applying $R_1 \to R_1 + R_2$

$A = \begin{vmatrix} x+y+z & x+y+z & x+y+z \\ z & x & y \\ 1 & 1 & 1 \end{vmatrix}$

$= (x+y+z) \begin{vmatrix} 1 & 1 & 1 \\ z & x & y \\ 1 & 1 & 1 \end{vmatrix}$

Since R_1 and R_3 are identical, thus

$A = (x + y + z). 0$

$\Rightarrow \quad A = 0$

or $\begin{vmatrix} x+y & y+z & z+x \\ z & x & y \\ 1 & 1 & 1 \end{vmatrix} = 0$

Q. 13. Prove that :

$\begin{vmatrix} b+c & a-b & a \\ c+a & b-c & b \\ a+b & c-a & c \end{vmatrix} = 3abc - a^3 - b^3 - c^3.$ (*JAC*, 2015)

Solution : $\text{LHS} = \begin{vmatrix} b+c & a-b & a \\ c+a & b-c & b \\ a+b & c-a & c \end{vmatrix}$

Applying $R_1 \to R_1 + R_2 + R_3$

$= \begin{vmatrix} 2(a+b+c) & 0 & a+b+c \\ c+a & b-c & b \\ a+b & c-a & c \end{vmatrix}$

$= (a+b+c) \begin{vmatrix} 2 & 0 & 1 \\ c+a & b-c & b \\ a+b & c-a & c \end{vmatrix}$

Applying $C_1 \to C_1 - 2C_3$

$= (a+b+c) \begin{vmatrix} 0 & 0 & 1 \\ (c+a-2b) & b-c & b \\ (a+b-2c) & c-a & c \end{vmatrix}$

$$= (a+b+c).1 \begin{vmatrix} c+a-2b & b-c \\ a+b-2c & c-a \end{vmatrix} \quad \text{(Expanding } C_3\text{)}$$

Applying $C_1 \to C_2 + 2C_2$

$$= (a+b+c) \begin{vmatrix} a-c & b-c \\ b-a & c-a \end{vmatrix}$$

$$= (a+b+c)\,[(a-c)(c-a)-(b-a)(b-c)]$$
$$= (a+b+c)\,[ac-a^2-c^2+ac-b^2+bc+ab-ac]$$
$$= (a+b+c)(ac+bc+ab-a^2-b^2-c^2)$$
$$= (a+b+c)\,[-(a^2+b^2+c^2-ab-bc-ca)]$$
$$= 3abc-a^3-b^3-c^3 = \text{RHS}$$

Q. 14. Solve the following equations by matrix method :

$$x-2y+z=0$$
$$2x-y+z=3$$
$$x+y+z=6$$

(*JAC*, 2015)

Solution : Here $A = \begin{vmatrix} 1 & -2 & 1 \\ 2 & -1 & 1 \\ 1 & 1 & 1 \end{vmatrix}$, $X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ and $B = \begin{bmatrix} 0 \\ 3 \\ 6 \end{bmatrix}$

$$\therefore \quad |A| = \begin{vmatrix} 1 & -2 & 1 \\ 2 & -1 & 1 \\ 1 & 1 & 1 \end{vmatrix} = 1(-1-1)+2(2-1)+1(2+1)$$
$$= -2+2+3$$
$$= 3 \neq 0$$

$$\therefore \quad A_{11} = (-1)^{1+1} \begin{vmatrix} -1 & 1 \\ 1 & 1 \end{vmatrix} = 1 \times (-1-1) = -2$$

$$A_{12} = (-1)^{1+2} \begin{vmatrix} 2 & 1 \\ 1 & 1 \end{vmatrix} = -1 \times (2-1) = -1$$

$$A_{13} = (-1)^{1+3} \begin{vmatrix} 2 & -1 \\ 1 & 1 \end{vmatrix} = 1 \times (2+1) = 3$$

$$A_{21} = (-1)^{2+1} \begin{vmatrix} -2 & 1 \\ 1 & 1 \end{vmatrix} = -1 \times (-2-1) = 3$$

$$A_{22} = (-1)^{2+2} \begin{vmatrix} 1 & 1 \\ 1 & 1 \end{vmatrix} = 1 \times (1-1) = 0$$

$$A_{23} = (-1)^{2+3} \begin{vmatrix} 1 & -2 \\ 1 & 1 \end{vmatrix} = -1 \times (1+2) = -3$$

$$A_{31} = (-1)^{3+1} \begin{vmatrix} -2 & 1 \\ -1 & 1 \end{vmatrix} = 1 \times (-2+1) = -1$$

$$A_{32} = (-1)^{3+2} \begin{vmatrix} 1 & 1 \\ 2 & 1 \end{vmatrix} = -1 \times (1-2) = 1$$

$$A_{33} = (-1)^{3+3} \begin{vmatrix} 1 & -2 \\ 2 & -1 \end{vmatrix} = 1 \times (-1+4) = 3$$

$$\therefore adj.\,A = \begin{bmatrix} -2 & -1 & 3 \\ 3 & 0 & -3 \\ -1 & 1 & 3 \end{bmatrix}' = \begin{bmatrix} -2 & 3 & -1 \\ -1 & 0 & 1 \\ 3 & -3 & 3 \end{bmatrix}$$

$$\therefore \quad A^{-1} = \frac{adj\,A}{|A|} = \frac{1}{3} \begin{bmatrix} -2 & 3 & -1 \\ -1 & 0 & 1 \\ 3 & -3 & 3 \end{bmatrix}$$

$$\because \quad AX = B$$
$$\Rightarrow \quad A^{-1}(AX) = A^{-1}B$$

$$\therefore \quad \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \frac{1}{3} \begin{bmatrix} -2 & 3 & -1 \\ -1 & 0 & 1 \\ 3 & -3 & 3 \end{bmatrix} \begin{bmatrix} 0 \\ 3 \\ 6 \end{bmatrix}$$

$$= \frac{1}{3} \begin{bmatrix} 0+9-6 \\ 0+0+6 \\ 0-9+18 \end{bmatrix} = \frac{1}{3} \begin{bmatrix} 3 \\ 6 \\ 9 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

$$\therefore \quad x = 1, y = 2 \text{ and } z = 3$$

Q. 15. Prove that :

$$\begin{bmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{bmatrix} = (a-b)(b-c)(c-a)$$

(*USEB*, 2015)

Solution : $\quad \text{LHS} = \begin{bmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{bmatrix}$

Applying $R_2 \to R_2 - R_1$ and $R_3 \to R_3 - R_1$

$$= \begin{vmatrix} 1 & a & a^2 \\ 0 & b-a & b^2-a^2 \\ 0 & c-a & c^2-a^2 \end{vmatrix}$$

Taking common $(b-a)$ from R_2 and $(c-a)$ from R_3

$$= (b-a)(c-a) \begin{vmatrix} 1 & a & a^2 \\ 0 & 1 & b+a \\ 0 & 1 & c+a \end{vmatrix}$$

Applying $R_3 \to R_3 - R_2$

$$= (b-a)(c-a) \begin{vmatrix} 1 & a & a^2 \\ 0 & 1 & b+a \\ 0 & 0 & c-b \end{vmatrix}$$

Taking $(c-a)$ from R_3

$$= (b-a)(c-a)(c-b) \begin{vmatrix} 1 & a & a^2 \\ 0 & 1 & b+a \\ 0 & 0 & 1 \end{vmatrix}$$

Expanding C_1

$$= (b-a)(c-a)(c-b).1 \begin{vmatrix} 1 & b+a \\ 0 & 1 \end{vmatrix}$$
$$= (b-a)(c-a)(c-b).1$$
$$= (a-b)(b-c)(c-a)$$
$$= \text{RHS}$$

Q. 16. Evaluate the determinant :

$$\begin{bmatrix} 2 & 3 & 1 \\ 4 & 6 & 2 \\ 1 & 3 & 2 \end{bmatrix}$$

(*USEB*, 2015)

Solution : $\begin{bmatrix} 2 & 3 & 1 \\ 4 & 6 & 2 \\ 1 & 3 & 2 \end{bmatrix}$

Applying $R_2 \to R_2 - R_1$

$$= \begin{bmatrix} 2 & 3 & 1 \\ 2 & 3 & 1 \\ 1 & 3 & 2 \end{bmatrix}$$

Since R_1 and R_2 are identical, thus

$= 0$

NCERT QUESTIONS

Q. 1. Using properties of determinants, prove that :

$$\begin{vmatrix} -a^2 & ab & ac \\ ba & -b^2 & bc \\ ca & cb & -c^2 \end{vmatrix} = 4a^2b^2c^2$$ (*CBSE*, 2011)

Solution

Let $\Delta = \begin{vmatrix} -a^2 & ab & ac \\ ba & -b^2 & bc \\ ca & cb & -c^2 \end{vmatrix}$

Taking out common a, b, c from C_1, C_2, C_3 respectively

$$= abc \begin{vmatrix} -a & a & a \\ b & -b & b \\ c & c & -c \end{vmatrix}$$

Taking out common a, b, c from R_1, R_2, R_3 respectively

$$= a^2b^2c^2 \begin{vmatrix} -1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \end{vmatrix}$$

Applying $R_1 \to R_1 + R_2 + R_3$

$$= a^2b^2c^2 \begin{vmatrix} 0 & 0 & 2 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \end{vmatrix}$$

$$= a^2b^2c^2 [2(2)] = 4a^2b^2c^2$$

(Expanding along R_1)

Q. 2. Using properties of determinants, prove the following : (*CBSE, Delhi*, 2011)

$$\begin{vmatrix} x & y & z \\ x^2 & y^2 & z^2 \\ x^3 & y^3 & z^3 \end{vmatrix} = xyz\,(x-y)(y-z)(z-x).$$

Solution LHS $= \begin{vmatrix} x & y & z \\ x^2 & y^2 & z^2 \\ x^3 & y^3 & z^3 \end{vmatrix}$

Taking out common x, y, z from C_1, C_2, C_3

$$= xyz \begin{vmatrix} 1 & 1 & 1 \\ x & y & z \\ x^2 & y^2 & z^2 \end{vmatrix}$$

Applying $C_1 \to C_1 - C_2$ and $C_3 \to C_3 - C_2$

$$= xyz \begin{vmatrix} 0 & 1 & 0 \\ x-y & y & z-y \\ x^2-y^2 & y^2 & z^2-y^2 \end{vmatrix}$$

Taking out common $(x-y)$ and $(y-z)$ from C_1 and C_2 respectively

$$= xyz\,(x-y)\,(y-z) \begin{vmatrix} 0 & 1 & 0 \\ 1 & y & -1 \\ x+y & y^2 & -(y+z) \end{vmatrix}$$

Expanding along R_1

$$= xyz(x-y)(y-z)\,[-1\,(-y-z+x+y)]$$

$$= xyz(x-y)(y-z)(z-x) = \text{RHS}$$

Q. 3. Using properties of determinants, solve the following for x :

$$\begin{vmatrix} x-2 & 2x-3 & 3x-4 \\ x-4 & 2x-9 & 3x-16 \\ x-8 & 2x-27 & 3x-64 \end{vmatrix} = 0$$

(*CBSE Outside Delhi*, 2011)

Solution : $\begin{vmatrix} x-2 & 2x-3 & 3x-4 \\ x-4 & 2x-9 & 3x-16 \\ x-8 & 2x-27 & 3x-64 \end{vmatrix} = 0$

Applying $C_2 \to C_2 - 2C_1$ and $C_3 \to C_3 - 3C_1$

$$\Delta = \begin{vmatrix} x-2 & 1 & 2 \\ x-4 & -1 & -4 \\ x-8 & -11 & -40 \end{vmatrix} = 0$$

Applying $R_2 \to R_2 - R_1$ and $R_3 \to R_3 - R_1$

$$\Delta = \begin{vmatrix} x-2 & 1 & 2 \\ -2 & -2 & -6 \\ -6 & -12 & -42 \end{vmatrix}$$

Taking out common (– 2) and (– 6) from R_2 and R_3 respectively

$$\Delta = (-2)(-6) \begin{vmatrix} x-2 & 1 & 2 \\ 1 & 1 & 3 \\ 1 & 2 & 7 \end{vmatrix}$$

Applying $R_3 \to R_3 - R_2$

$$\Delta = 12 \begin{vmatrix} x-2 & 1 & 2 \\ 1 & 1 & 3 \\ 0 & 1 & 4 \end{vmatrix}$$

Expaning along C_1

$$\Delta = 12\,\{(x-2)(4-3) - (4-2)\} = 0$$

$$\Rightarrow \quad 12\,(x-4) = 0$$

$$\therefore \quad x = 4$$