$$\frac{dS}{dr} = -\frac{200}{r^2} + 4\pi r$$

$$\frac{d^2S}{dr^2} = \frac{400}{r^3} + 4\pi$$

$$\frac{dS}{dr} = 0$$

$$\Rightarrow \quad 4\pi r = \frac{200}{r^2}$$

$$\Rightarrow \quad 4\pi r^3 = 200$$

7 INDEFINITE INTEGRALS

IMPORTANT FORMULAE

- $\int x^n dx = \frac{x^{n+1}}{n+1}$ for $(n \neq -1)$

$\int \frac{1}{x} dx = \log |x|$ $\qquad$ $\int 1.dx = x$

$\int \sin x \, dx = -\cos x$ $\qquad$ $\int \cos x \, dx = \sin x$

$\int \sec^2 x \, dx = \tan x$ $\qquad$ $\int \text{cosec}^2 x \, dx = -\cot x$

$\int \sec x \tan x \, dx = \sec x$ $\qquad$ $\int \text{cosec}\, x \cot x \, dx = -\text{cosec}\, x$

$\int e^x dx = e^x$ $\qquad$ $\int a^x dx = \frac{a^x}{\log_e a}$

$\int \frac{1}{\sqrt{1-x^2}} dx = \sin^{-1} x$ $\qquad$ $\int \frac{1}{\sqrt{1+x^2}} dx = \tan^{-1} x$

$\int \frac{1}{x\sqrt{x^2-1}} dx = \sec^{-1} x$ $\qquad$ $\int \tan x \, dx = \log |\sec x|$

$\int \cot x \, dx = \log |\sin x|$

$\int \sec x \, dx = \log |\sec x + \tan x|$

$\int \text{cosec}\, x \, dx = \log |\text{cosec}\, x - \cot x|$

$\int \frac{f'(x)}{f(x)} dx = \log f(x)$ $\qquad$ $\int \frac{1}{x^2 - a^2} dx = \frac{1}{2a} \log \left|\frac{x-a}{x+a}\right|$

$\int \frac{1}{a^2 - x^2} dx = \frac{1}{2a} \log \left|\frac{a+x}{a-x}\right|$

$\int \frac{1}{a^2 + x^2} dx = \frac{1}{a} \tan^{-1}\left(\frac{x}{a}\right)$ $\qquad$ $\int \frac{1}{\sqrt{a^2 - x^2}} dx = \sin^{-1}\left(\frac{x}{a}\right)$

$\int \frac{1}{\sqrt{a^2+x^2}} dx = \log |x + \sqrt{a^2+x^2}|$

$\int \frac{1}{\sqrt{x^2-a^2}} dx = \log |x + \sqrt{x^2-a^2}|$

$\int \sqrt{a^2 - x^2} \, dx = \frac{x}{2}\sqrt{a^2-x^2} + \frac{a^2}{2} \sin^{-1}\left(\frac{x}{a}\right)$

$\int \sqrt{a^2 + x^2} \, dx = \frac{x}{2}\sqrt{a^2+x^2} + \frac{a^2}{2} \log |x + \sqrt{a^2+x^2}|$

$\int \sqrt{x^2 - a^2} \, dx = \frac{x}{2}\sqrt{x^2-a^2} - \frac{a^2}{2} \log |x + \sqrt{x^2-a^2}|$

- $\int [f(x) \pm g(x)] \, dx = \int f(x) \, dx \pm \int g(x) \, dx$
- $\int c.f(x) \, dx = c.\int f(x) \, dx$
- $\int f(x).g(x) dx = f(x).\int g(x) dx - \int \left(\frac{d}{dx} f(x)\right)\left(\int g(x) dx\right) dx$

Partial fractions

$\frac{f(x)}{(x-a)(x-b)} = \frac{A}{x-a} + \frac{B}{x-b}$ [degree of $f(x) \leq 1$]

$\frac{f(x)}{(x-a)(x-b)(x-c)} = \frac{A}{x-a} + \frac{B}{x-b} + \frac{C}{x-c}$ [degree of $f(x) \leq 2$]

$\frac{f(x)}{(x-a)^2(x-b)} = \frac{A}{x-a} + \frac{B}{(x-a)^2} + \frac{C}{x-b}$ [degree of $f(x) \leq 2$]

$\frac{f(x)}{(x-a)^3(x-b)} = \frac{A}{x-a} + \frac{B}{(x-a)^2} + \frac{C}{(x-a)^3} + \frac{D}{x-b}$ [degree of $f(x) \leq 3$]

$\frac{f(x)}{(x-d)(ax^2+bx+c)} = \frac{A}{x-d} + \frac{Bx+C}{ax^2+bx+c}$ [degree of $f(x) \leq 2$]

- To finding the value of

$\int \frac{dx}{a + b\sin^2 x + c\cos^2 x + d \sin x \cos x}$ Divide by $\cos^2 x$ in numerator and denominator and let $\tan x = t$.

Multiple Choice Questions

1. $\int \frac{(1+\log x)^2}{x} dx =$ (*BSEB*, 2010)

(a) $\frac{1}{3}(1+\log x)^3 + C$ $\qquad$ (b) $\frac{1}{2}(1+\log x)^2 + C$

(c) $\log(1+\log x)$ $\qquad$ (d) none of these

2. $\int \frac{\cos 2x}{(\sin x + \cos x)^2} dx =$ (*BSEB*, 2011)

(a) $\frac{1}{\sin x + \cos x} + C$ $\qquad$ (b) $\log(\sin x + \cos x) + C$

(c) $\log |\sin x - \cos x| + C$ $\qquad$ (d) $\frac{1}{(\sin x + \cos x)^2}$

3. $\int \frac{1}{1+\sin x}\,dx =$

(a) $\tan x - \sec x + C$ (b) $\sec x - \tan x + C$
(c) $\sec x + \tan x + C$ (d) none of these

4. $\int \frac{xe^x}{(1+x)^2}\,dx =$

(a) $\frac{e^x}{x+1} + C$ (b) $e^x(x+1) + C$
(c) $-\frac{e^x}{(x+1)^2} + C$ (d) $\frac{e^x}{1+x^2} + C$

5. $\int \frac{1}{x^2(x^4+1)^{3/4}}\,dx =$

(a) $-\left(1+\frac{1}{x^4}\right)^{1/4} + C$ (b) $(x^4+1)^{1/4} + C$
(c) $\left(1-\frac{1}{x^4}\right)^{1/4} + C$ (d) $-\left(1+\frac{1}{x^4}\right)^{3/4} + C$

6. $\int \frac{1-\sin x}{\cos^2 x}\,dx =$

(a) $\tan x - \sec x + C$ (b) $\tan x + \sec x + C$
(c) $\sec x - \tan x + C$ (d) none of these

7. $\int \log x\ dx =$

(a) $x(\log x - 1) + C$ (b) $x(\log x + 1) + C$
(c) $x\log x + C$ (d) $x(1 - \log x) + C$

8. $\int \frac{1}{x^2+2x+2}\,dx =$

(a) $\tan^{-1}(x+1) + C$ (b) $\tan^{-1}(x+2) + C$
(c) $\sin^{-1}(x+1) + C$ (d) $\sin^{-1}(x+2) + C$

9. $\int \frac{10x^9 + 10^x \log_e 10}{x^{10}+10^x}\,dx =$

(a) $(x^{10}+10^x)^{-1} + C$ (b) $10^x - x^{10} + C$
(c) $x^{10} + 10^x + C$ (d) $\log(x^{10}+10^x) + C$

10. $\int x^2 e^{x^3} \cos\left(e^{x^3}\right) dx =$

(a) $\sin\left(e^{x^3}\right) + C$ (b) $\frac{1}{3}\sin\left(e^{x^3}\right) + C$
(c) $-\frac{1}{3}\sin\left(e^{x^3}\right) + C$ (d) $3\sin\left(e^{x^3}\right) + C$

11. $\int 1.dx =$ (*BSEB*, 2015)

(a) $x + k$ (b) $1 + k$
(c) $\frac{x^2}{2} + k$ (d) $\log x + k$

12. $\int \frac{dx}{\sqrt{x}} =$ (*BSEB*, 2015)

(a) $\sqrt{x} + k$ (b) $2\sqrt{x} + k$
(c) $x + k$ (d) $\frac{2}{3}x^{3/2} + k$

13. If $x > a$ $\int \frac{dx}{x^2 - a^2} =$ (*BSEB*, 2015)

(a) $\frac{1}{2a}\log\frac{x-a}{x+a} + k$ (b) $\frac{1}{2a}\log\frac{x+a}{x-a} + k$
(c) $\frac{1}{a}\log(x^2-a^2) + k$ (d) $\log\left(x+\sqrt{x^2-a^2}\right) + k$

Ans. 1. (a), 2. (b), 3. (a), 4. (a), 5. (a), 6. (a), 7. (a), 8. (a), 9. (d), 10. (b), 11. (a), 12. (b), 13. (a).

➠ Very Short Answer Type Questions

Q. 1. Evaluate : $\int \frac{\sec^2(\log x)}{x}\,dx$. (*BSEB*, 2013)

Solution : $I = \int \frac{\sec^2(\log x)}{x}\,dx$

(Put $\log x = t$; $\therefore \frac{1}{x}dx = dt$)

$\therefore \quad I = \int \sec^2 t\,dt$

$\Rightarrow \quad I = \tan t + C$

$\Rightarrow \quad I = \tan(\log x) + C$

Q. 2. $\int \frac{d}{dx}(\log_e^x)\,dx =$ .... $+ k$, **where** k **is a constant.** (*BSEB*, 2014)

Solution : $I = \int \frac{d}{dx}(\log_e^x)\,dx$

$= \log_e^x + k$, where k is a constant.

Q. 3. Write the value of $\int \sec x\,dx$. (*BSEB*, 2014)

Solution : $\int \sec x\,dx = \log(\sec x + \tan x) + C$

Q. 4. Find anti-derivative of $\left(\sqrt{x} + \frac{1}{\sqrt{x}}\right)$. (*USEB*, 2014)

Solution : $I = \int\left(\sqrt{x} + \frac{1}{\sqrt{x}}\right) dx$

$= \int \sqrt{x}\,dx + \int x^{-\frac{1}{2}}\,dx$

$= \frac{x^{\frac{1}{2}+1}}{\frac{1}{2}+1} + \frac{x^{-\frac{1}{2}+1}}{-\frac{1}{2}+1} + C$

$= \frac{2}{3}x^{3/2} + 2\sqrt{x} + C$

Q. 5. Write the anti-derivative of $\left(3\sqrt{x} + \frac{1}{\sqrt{x}}\right)$. (*CBSE*, 2014)

Solution : $I = \int\left(3\sqrt{x} + \frac{1}{\sqrt{x}}\right) dx$

$= 3\int \sqrt{x}\,dx + \int x^{-1/2}\,dx$

$$= 3\frac{x^{3/2}}{3/2} + \frac{x^{1/2}}{1/2} + C$$

$$= 2x^{3/2} + 2x^{1/2} + C$$

Q. 6. Evaluate : $\int \frac{\sin^6 x}{\cos^8 x}\,dx.$

[*AI CBSE*, 2014 (*Comptt.*)]

Solution : $I = \int \frac{\sin^6 x}{\cos^8 x}\,dx$

$$= \int \tan^6 x \sec^2 x\,dx$$

(Put $\tan x = t$; $\therefore \sec^2 x\,dx = dt$)

$$= \int t^6\,dt$$

$$= \frac{t^7}{7} + C$$

$$= \frac{\tan^7 x}{7} + C$$

Q. 7. Find : $\int \frac{\sin^2 x - \cos^2 x}{\sin^2 x \cos^2 x}\,dx.$

[*CBSE*, 2014 (*Comptt.*)]

Solution : $I = \int \frac{\sin^2 x - \cos^2 x}{\sin^2 x \cos^2 x}\,dx$

$$= \int (\sec^2 x - \operatorname{cosec}^2 x)\,dx$$

$$= \tan x + \cot x + C$$

Q. 8. Evaluate : $\int \frac{e^{2x} - 1}{e^{2x} + 1}\,dx.$ (*BSER*, 2014)

Solution : $I = \int \frac{e^{2x} - 1}{e^{2x} + 1}\,dx$

$$= \int \frac{e^x - e^{-x}}{e^x + e^{-x}}\,dx$$

[Put $e^x + e^{-x} = t \Rightarrow (e^x - e^{-x})\,dx = dt$]

$\therefore$ $I = \int \frac{dt}{t}$

$$= \log t + C$$

$$= \log(e^x + e^{-x}) + C$$

Q. 9. Evaluate : $\int \frac{1}{x^2}\,dx.$ (*JAC*, 2013)

Solution : $I = \int \frac{1}{x^2}\,dx$

$$= \frac{x^{-2+1}}{-2+1} + C$$

$$= -\frac{1}{x} + C$$

Q. 10. Evaluate : $\int \sqrt{5 - 4x - x^2}\,dx.$ (*BSER*, 2013)

Solution : $I = \int \sqrt{5 - 4x - x^2}\,dx$

$$= \int \sqrt{5 - (x^2 + 4x)}\,dx$$

$$= \int \sqrt{5 - (x^2 + 4x + 4) + 4}\,dx$$

$$= \int \sqrt{9 - (x^2 + 4x + 4)}\,dx$$

$$= \int \sqrt{3^2 - (x + 2)^2}\,dx$$

$$= \frac{1}{2}(x + 2)\sqrt{3^2 - (x + 2)^2} + \frac{1}{2}.3^2 \sin^{-1}\left(\frac{x + 2}{3}\right) + C$$

$$= \frac{1}{2}(x + 2)\sqrt{5 - 4x - x^2} + \frac{9}{2}\sin^{-1}\left(\frac{x + 2}{3}\right) + C$$

Q. 11. Evaluate : $\int \frac{e^{\tan^{-1} x}}{(1 + x^2)}\,dx.$ (*BSEB*, 2014)

Solution : $I = \int \frac{e^{\tan^{-1} x}}{(1 + x^2)}\,dx$

(Put $\tan^{-1} x = 1$, $\therefore \frac{1}{1 + x^2}\,dx = dt$)

$\therefore$ $I = \int e^t\,dt$

$\Rightarrow$ $I = e^t + C$

$\Rightarrow$ $I = e^{\tan^{-1} x} + C$

Q. 12. Evaluate : $\int a^{3 \log a^x}\,dx.$ (*BSER*, 2013)

Solution : $I = \int a^{3 \log a^x}\,dx$

$$= \int a^{3x \log a}\,dx$$

$$= \int a^{3 \log ax}\,dx$$

$= \int a^{Ax}\,dx$, where $A = 3 \log a$

$$= \int a^t \frac{dt}{A} = \frac{1}{A}\int a^t\,dt$$

(Put $Ax = t$; $\therefore A\,dx = dt$; $\therefore dx = \frac{dt}{A}$)

$$= \frac{1}{A}\frac{a^t}{\log a} + C$$

$$= \frac{a^t}{3(\log a)^2} + C$$

$$= \frac{a^{Ax}}{3(\log a)^2} + C$$

$$= \frac{a^{3 \log ax}}{3(\log a)^2} + C$$

$$= \frac{a^{3x \log a}}{3(\log a)^2} + C$$

$$= \frac{a^{3 \log a^x}}{3(\log a)^2} + C$$

Q. 13. Evaluate : $\int \frac{1}{1+\cos x}\,dx$. (*JAC*, 2013)

Solution :

$$I = \int \frac{1}{1+\cos x}\,dx$$

$$= \int \frac{1}{2\cos^2 \frac{x}{2}}\,dx$$

$$= \frac{1}{2}\int \sec^2 \frac{x}{2}\,dx$$

$$\left(\text{Put } \frac{x}{2} = t,\ \therefore\ \frac{1}{2}dx = dt\right)$$

$$= \int \sec^2 t\,dt$$

$$= \tan t + C$$

$$= \tan \frac{x}{2} + C$$

Q. 14. Evaluate : $\int \tan^4 x\,dx$. (*JAC*, 2013)

Solution :

$$I = \int \tan^4 x\,dx$$

$$= \int \tan^2 x \tan^2 x\,dx$$

$$= \int \tan^2 x\,(\sec^2 x - 1)\,dx$$

$$= \int \tan^2 x \sec^2 x\,dx - \int \tan^2 x\,dx$$

$$= \int \tan^2 x \sec^2 x\,dx - \int (\sec^2 x - 1)\,dx$$

$$= \frac{\tan^3 x}{3} - (\tan x - x) + C$$

(Putting $\tan x = t$ in I integral)

$$= \frac{1}{3}\tan^3 x - \tan x + x + C$$

Q. 15. Evaluate : $\int \frac{\cos(\log x)}{x}\,dx$. (*JAC*, 2014)

Solution :

$$I = \int \frac{\cos(\log x)}{x}$$

$$\left(\text{Put } \log x = t,\ \therefore\ \frac{1}{x}\,dx = dt\right)$$

$$= \int \cos t\,dt$$

$$= \sin t + C$$

$$= \sin(\log x) + C$$

Q. 16. Evaluate : $\int \frac{\cos 2x - \cos 2\alpha}{\cos x - \cos \alpha}\,dx$.

(*AI CBSE*, 2013)

Solution :

$$I = \int \frac{\cos 2x - \cos 2\alpha}{\cos x - \cos\alpha}\,dx$$

$$= \int \frac{(2\cos^2 x - 1) - (2\cos^2\alpha - 1)}{\cos x - \cos\alpha}\,dx$$

$$= 2\int \frac{(\cos x - \cos\alpha)(\cos x + \cos\alpha)}{\cos x - \cos\alpha}\,dx$$

$$= 2\int (\cos x + \cos\alpha)\,dx$$

$$= 2(\sin x + x\cos\alpha) + C$$

Q. 1. Evaluate : $\int \frac{\sin^6 x + \cos^6 x}{\sin^2 x \cos^2 x}\,dx$. (*CBSE*, 2014)

Solution :

$$I = \int \frac{\sin^6 x + \cos^6 x}{\sin^2 x\cos^2 x}\,dx$$

$$= \int \frac{(\sin^2 x + \cos^2 x)^3 - 3\sin^2 x\cos^2 x(\sin^2 x + \cos^2 x)}{\sin^2 x\cos^2 x}\,dx$$

$$= \int \frac{1 - 3\sin^2 x\cos^2 x}{\sin^2 x\cos^2 x}\,dx$$

$$= \int \left(\frac{1}{\sin^2 x\cos^2 x} - 3\right)dx$$

$$= \int \left(\frac{\sin^2 x + \cos^2 x}{\sin^2 x\cos^2 x} - 3\right)dx$$

$$= \int (\sec^2 x + \operatorname{cosec}^2 x - 3)\,dx$$

$$= \tan x - \cot x - 3x + C$$

Q. 2. Evaluate : $\int \frac{dx}{\sqrt{\sin^3 x \sin(x+a)}}$. (*BSER*, 2014)

Solution :

$$I = \int \frac{dx}{\sqrt{\sin^3 x\sin(x+a)}}$$

$$= \int \frac{\sqrt{\sin x}}{\sqrt{\sin^4 x\sin(x+a)}}\,dx$$

$$= \int \frac{1}{\sin^2 x}\,dx\,\sqrt{\frac{\sin x}{\sin(x+a)}}\,dx$$

$$\text{Put } \frac{\sin(x+a)}{\sin x} = t$$

$$\therefore \quad \frac{\sin x\cos(x+a) - \sin(x+a)\cos x}{\sin^2 x}\,dx = dt$$

$$\Rightarrow \quad -\frac{\sin a}{\sin^2 x}\,dx = dt$$

$$\Rightarrow \quad \frac{1}{\sin^2 x}\,dx = -\frac{1}{\sin a}\,dt$$

$$\therefore \quad I = -\frac{1}{\sin a}\int \frac{1}{\sqrt{t}}\,dt$$

$$= -\frac{1}{\sin a}\,2\sqrt{t} + C$$

$$= -\frac{2}{\sin a}\sqrt{\frac{\sin(x+a)}{\sin x}} + C$$

Q. 3. Evaluate : $\int \frac{e^x}{\sqrt{5 - 4e^x - e^{2x}}}\,dx$. (*JAC*, 2014)

Solution :

$$I = \int \frac{e^x}{\sqrt{5 - 4e^x - e^{2x}}}\,dx$$

$$(\text{Put } e^x = t,\ \therefore\ e^x dx = dt)$$

$$= \int \frac{dt}{\sqrt{5 - 4t - t^2}}$$

$$= \int \frac{dt}{\sqrt{5 - (t^2 + 4t + 4) + 4}}$$

$$= \int \frac{dt}{\sqrt{9 - (t + 2)^2}}$$

$$= \int \frac{dt}{\sqrt{3^2 - (t + 2)^2}}$$

$$= \sin^{-1}\left(\frac{t + 2}{3}\right) + C$$

$$= \sin^{-1}\left(\frac{e^x + 2}{3}\right) + C$$

Q. 4. Integrate the function $\dfrac{\cos x}{(1 - \sin x)(2 - \sin x)}$ **with respect to** x. *(USEB, 2013)*

Solution : $I = \dfrac{\cos x}{(1 - \sin x)(2 - \sin x)} dx$

(**Put** $\sin x = t$, $\therefore \cos x \, dx = dt$)

$$\therefore \quad I = \int \frac{dt}{(1 - t)(2 - t)}$$

$$= \int \left(\frac{1}{1 - t} - \frac{1}{2 - t}\right) dt$$

(Resolving integrand into partial fraction)

$$= -\log(1 - t) + \log(2 - t) + C$$

$$= \log \frac{2 - t}{1 - t} + C$$

$$= \log \frac{2 - \sin x}{1 - \sin x} + C$$

Q. 5. Evaluate : $\displaystyle\int \frac{\sin(x - a)}{\sin(x + a)} dx$. *(CBSE, 2013)*

Solution : $I = \displaystyle\int \frac{\sin(x - a)}{\sin(x + a)} dx$

(**Put** $x + a = t \Rightarrow x = t - a$, $\therefore dx = dt$)

$$= \int \frac{\sin(t - 2a)}{\sin t} dt$$

$$= \int \frac{\sin t \cos 2a - \cos t \sin 2a}{\sin t} dt$$

$$= \int (\cos 2a - \sin 2a \cot t) \, dt$$

$$= \cos 2a \int dt - \sin 2a \int \cot t \, dt$$

$$= \cos 2a \, (t) - \sin 2a \log \sin t + C$$

$$= (x + a) \cos 2a - \sin 2a \log \sin (x + a) + C$$

Q. 6. Evaluate : $\displaystyle\int \frac{x^2}{(x^2 + 4)(x^2 + 9)} dx$. *(CBSE, 2013)*

Solution : $I = \displaystyle\int \frac{x^2}{(x^2 + 4)(x^2 + 9)} dx$

$$= \int \left(-\frac{4}{5} \frac{1}{x^2 + 4} + \frac{9}{5} \frac{1}{x^2 + 9}\right) dx$$

(Resolving the integrand into partial fraction)

$$= -\frac{4}{5} \int \frac{1}{x^2 + 2^2} dx + \frac{9}{5} \int \frac{1}{x^2 + 3^2} dx$$

$$= -\frac{4}{5} \cdot \frac{1}{2} \tan^{-1}\left(\frac{x}{2}\right) + \frac{9}{5} \cdot \frac{1}{3} \tan^{-1}\left(\frac{x}{3}\right) + C$$

$$= -\frac{2}{5} \tan^{-1}\left(\frac{x}{2}\right) + \frac{3}{5} \tan^{-1}\left(\frac{x}{3}\right) + C$$

Q. 7. Evaluate : $\displaystyle\int \frac{x^2 + 1}{(x^2 + 4)(x^2 + 25)} dx$.

(CBSE, 2013)

Solution : $I = \displaystyle\int \frac{x^2 + 1}{(x^2 + 4)(x^2 + 25)} dx$

$$= \int \left(-\frac{1}{7} \frac{1}{x^2 + 4} + \frac{8}{7} \frac{1}{x^2 + 25}\right) dx$$

(Resolving the integrand into partial fraction)

$$= -\frac{1}{7} \int \frac{1}{x^2 + 2^2} dx + \frac{8}{7} \int \frac{1}{x^2 + 5^2} dx$$

$$= -\frac{1}{7} \cdot \frac{1}{2} \tan^{-1}\left(\frac{x}{2}\right) + \frac{8}{7} \cdot \frac{1}{5} \tan^{-1}\left(\frac{x}{5}\right) + C$$

$$= -\frac{1}{14} \tan^{-1}\left(\frac{x}{2}\right) + \frac{8}{35} \tan^{-1}\left(\frac{x}{5}\right) + C$$

Q. 8. Evaluate : $\displaystyle\int \frac{2x^2 + 1}{x^2 (x^2 + 4)} dx$. *(CBSE, 2013)*

Solution : $I = \displaystyle\int \frac{2x^2 + 1}{x^2 (x^2 + 4)} dx$

$$= \int \left(\frac{1}{4} \cdot \frac{1}{x^2} + \frac{7}{4} \cdot \frac{1}{x^2 + 4}\right) dx$$

(Resolving the integrand into partial fraction)

$$= \frac{1}{4} \int \frac{1}{x^2} dx + \frac{7}{4} \int \frac{1}{x^2 + 2^2} dx$$

$$= -\frac{1}{4x} + \frac{7}{4} \cdot \frac{1}{2} \tan^{-1}\left(\frac{x}{2}\right) + C$$

$$= -\frac{1}{4x} + \frac{7}{8} \tan^{-1}\left(\frac{x}{2}\right) + C$$

Q. 9. Evaluate : $\displaystyle\int \frac{dx}{x(x^5 + 3)}$. *(AI CBSE, 2013)*

Solution : $I = \displaystyle\int \frac{dx}{x(x^5 + 3)}$

$$= \int \frac{x^4}{x^5 (x^5 + 3)} dx$$

(Put $x^5 = t$, $\therefore 5x^4 \, dx = dt$, $\therefore x^4 \, dx = \frac{1}{5} dt$)

$$= \frac{1}{5} \int \frac{dt}{t(t + 3)}$$

$$= \frac{1}{5} \cdot \frac{1}{3} \int \left(\frac{1}{t} - \frac{1}{t + 3}\right) dt$$

$$= \frac{1}{15} \{\log t - \log(t + 3)\} + C$$

$$= \frac{1}{15} \log \frac{t}{t+3} + C$$

$$= \frac{1}{15} \log \frac{x^5}{x^5+3} + C$$

Q. 10. Evaluate : $\int \frac{dx}{x(x^3+8)}$. (*AI CBSE*, 2013)

Solution : $I = \int \frac{dx}{x(x^3+8)} dx$

$$= \int \frac{x^2}{x^3(x^3+8)} dx$$

(Put $x^3 = t$, $\therefore 3x^2\,dx = dt$, $\therefore x^2\,dx = \frac{1}{3}dt$)

$$= \frac{1}{3}\int \frac{dt}{t(t+8)}$$

$$= \frac{1}{3}\cdot\frac{1}{8}\int\left(\frac{1}{t} - \frac{1}{t+8}\right)dt + C$$

$$= \frac{1}{24}\{\log t - \log(t+8)\} + C$$

$$= \frac{1}{24} \log \frac{t}{t+8} + C$$

$$= \frac{1}{24} \log \frac{x^3}{x^3+8} + C$$

$$= \frac{1}{8} \log \frac{x^3}{(x^3+8)^{1/3}} + C$$

Q. 11. Evaluate : $\int \frac{dx}{x(x^3+1)}$. (*AI CBSE*, 2013)

Solution : $I = \int \frac{dx}{x(x^3+1)} dx$

$$= \int \frac{x^2}{x^3(x^3+1)} dt$$

(Put $x^3 = t$, $\Rightarrow 3x^2\,dx = dt \Rightarrow x^2\,dx = \frac{dt}{3}$)

$$= \frac{1}{3}\int \frac{dt}{t(t+1)}$$

$$= \frac{1}{3}\int\left(\frac{1}{t} - \frac{1}{t+1}\right)dt$$

$$= \frac{1}{3}\{\log t - \log(t+1)\} + C$$

$$= \frac{1}{3} \log\left(\frac{t}{t+1}\right) + C$$

$$= \frac{1}{3} \log\left(\frac{x^3}{x^3+1}\right) + C$$

$$= \log \frac{x}{(x^3+1)^{1/3}} + C$$

Q. 12. Evaluate : $\int \frac{x^2}{(x^2+1)(x^2+4)} dx$.

[*CBSE*, 2013, 14 (*Comptt.*)]

Solution : $I = \int \frac{x^2}{(x^2+1)(x^2+4)} dx$

$$= \int\left(-\frac{1}{3}\cdot\frac{1}{x^2+1} + \frac{4}{3}\cdot\frac{1}{x^2+2^2}\right)dx$$

(Resolving the integrand into partial fraction)

$$= -\frac{1}{3}\tan^{-1} x + \frac{4}{3}\cdot\frac{1}{2}\tan^{-1}\frac{x}{2} + C$$

$$= -\frac{1}{3}\tan^{-1} x + \frac{2}{3}\tan^{-1}\frac{x}{2} + C$$

Q. 13. Evaluate : $\int \frac{x^2+x+1}{(x+2)(x^2+1)} dx$.

[*CBSE*, 2013 (*Comptt.*)]

Solution : $I = \int \frac{x^2+x+1}{(x+2)(x^2+1)} dx$

$$= \int \frac{1}{5}\left\{\frac{3}{x+2} + \frac{2x+1}{x^2+1}\right\} dx$$

(Resolving the integrand into partial fraction)

$$= \frac{1}{5}\int\left\{\frac{3}{x+2} + \frac{2x}{x^2+1} + \frac{1}{x^2+1}\right\} dx$$

$$= \frac{1}{5}[3\log(x+2) + \log(x^2+1) + \tan^{-1} x] + C$$

Q. 14. Evaluate : $\int \frac{8}{(x+2)(x^2+4)} dx$.

[*CBSE*, 2013 (*Comptt.*)]

Solution :

$$I = \int \frac{8}{(x+2)(x^2+4)} dx$$

$$= \int\left(\frac{1}{x+2} + \frac{-x+2}{x^2+4}\right)dx$$

(Resolving the integrand into partial fraction)

$$= \int \frac{1}{x+2}dx - \frac{1}{2}\int\frac{2x}{x^2+4}dx + 2\int\frac{1}{x^2+2^2}dx$$

$$= \log(x+2) - \frac{1}{2}\log(x^2+4) + \tan^{-1}\frac{x}{2} + C$$

Q. 15. Evaluate : $\int \sin x \sin 2x \sin 3x \, dx$.

(*CBSE, Delhi*, 2012)

Solution : We have

$$\int \sin x \sin 2x \sin 3x \, dx$$

$$= \frac{1}{2}\int(2\sin x \sin 3x)\sin 2x\,dx$$

$$= \frac{1}{2}\int(-\cos 4x + \cos 2x)\sin 2x\,dx$$

$$= \frac{1}{2}\int(-2\cos^2 2x + 1 + \cos 2x)\sin 2x\,dx$$

$$= -\int \cos^2 2x \sin 2x\,dx + \frac{1}{2}\int \sin 2x\,dx$$

$$+ \frac{1}{2}\int \cos 2x \sin 2x\, dx$$

$$= \frac{1}{2}\frac{\cos^3 2x}{3} - \frac{1}{2.2}\cos 2x - \frac{1}{2.2}\frac{\cos^2 2x}{2} + C$$

$$= \frac{1}{6}\cos^3 2x - \frac{1}{4}\cos 2x - \frac{1}{8}\cos^2 2x + C$$

Q. 16. Find $\int \frac{\sin^{-1}\sqrt{x} - \cos^{-1}\sqrt{x}}{\sin^{-1}\sqrt{x} + \cos^{-1}\sqrt{x}}\, dx;\ x \in [0, 1].$

[AI CBSE, 2014 (Comptt.)]

Solution :

$$I = \int \frac{\sin^{-1}\sqrt{x} - \cos^{-1}\sqrt{x}}{\sin^{-1}\sqrt{x} + \cos^{-1}\sqrt{x}}\, dx$$

$$= \int \frac{\sin^{-1}\sqrt{x} - \left(\frac{\pi}{2} - \sin^{-1}\sqrt{x}\right)}{\pi/2}\, dx$$

$$(\because \sin^{-1}\theta + \cos^{-1}\theta = \frac{\pi}{2})$$

$$= \frac{2}{\pi}\int\left(2\sin^{-1}\sqrt{x} - \frac{\pi}{2}\right)dx$$

$$= \frac{4}{\pi}\int \sin^{-1}\sqrt{x}\, dx - \int 1\, dx$$

$$= \frac{4}{\pi} I_1 - x + C$$

where $I_1 = \int \sin^{-1}\sqrt{x}\, dx$

(Put $x = \sin^2\theta \Rightarrow dx = 2\sin\theta\cos\theta\, d\theta$)

$$\therefore \quad I_1 = \int \sin^{-1}(\sin\theta)\, 2\sin\theta\cos\theta\, d\theta$$

$$= \int \theta \sin 2\theta\, d\theta$$

$$= \theta\left(\frac{-\cos 2\theta}{2}\right) - \int 1.\left(\frac{-\cos 2\theta}{2}\right)d\theta$$

$$= \frac{-\theta\cos 2\theta}{2} + \frac{\sin 2\theta}{4}$$

$$= -\frac{1}{2}\theta(1 - 2\sin^2\theta) + \frac{1}{4}(2\sin\theta\cos\theta)$$

$$= -\frac{1}{2}\sin^{-1}\sqrt{x}\,(1 - 2x) + \frac{1}{2}\sqrt{x}\cos\left(\sin^{-1}\sqrt{x}\right)$$

$$\therefore I = \frac{4}{\pi}\left[-\frac{1}{2}(1-2x)\sin^{-1}\sqrt{x} + \frac{1}{2}\sqrt{x}\cos\left(\sin^{-1}\sqrt{x}\right)\right] - x + C$$

Q. 17. Find $\int \frac{x\cos^{-1}(x)}{\sqrt{1 - x^2}}\, dx$. [AI CBSE, 2014 (Comptt.)]

Solution :

$$I = \frac{x\cos^{-1}(x)}{\sqrt{1-x^2}}\, dx$$

(Put $\cos^{-1}x = \theta \Rightarrow x = \cos\theta,\ \therefore\ -\frac{1}{\sqrt{1-x^2}}\, dx = d\theta$)

$$\therefore \quad I = -\int \underset{\text{II}}{\cos\theta}\,.\,\underset{\text{I}}{\theta}\, d\theta$$

$$= -[\theta\sin\theta - (1 \sin\theta\, d\theta)] + C$$

$$= -\theta\sin\theta - \cos\theta + C$$

$$= -\sqrt{1-x^2}\cos^{-1}x - x + C$$

$$= -\sqrt{1-x^2}\cos^{-1}x - x + C$$

Long Answer Type Questions

Q. 1. Evaluate : $\int (x - 3)\sqrt{x^2 + 3x - 18}\, dx$.

(CBSE, 2013)

Solution : $I = \int (x-3)\sqrt{x^2 + 3x - 18}\, dx$

Let $x - 3 = A\frac{d}{dx}(x^2 + 3x - 18) + B$

$\Rightarrow \quad x - 3 = A(2x + 3) + B$

$\Rightarrow \quad x - 3 = 2Ax + (3A + B)$

Comparing the coefficients, we get

$$2A = 1$$

$$3A + B = -3$$

$$\Rightarrow \quad A = \frac{1}{2},\ B = -\frac{9}{2}$$

$$\therefore \quad I = \frac{1}{2}\int \frac{d}{dx}(x^2 + 3x - 18)\sqrt{x^2 + 3x - 18}\, dx - \frac{9}{2}\int \sqrt{x^2 + 3x - 18}\, dx$$

$$= \frac{1}{2}I_1 - \frac{9}{2}I_2$$

where $I_1 = \int \frac{d}{dx}(x^2 + 3x - 18).\sqrt{x^2 + 3x - 18}\, dx$

$$= \int (2x+3)\sqrt{x^2 + 3x - 18}\, dx$$

(Put $x^2 + 3x - 18 = t \Rightarrow (2x + 3)\, dx = dt$)

$$\therefore \quad I_1 = \int \sqrt{t}\, dt$$

$$= \frac{t^{3/2}}{3/2} + C_1$$

$$= \frac{2}{3}t^{3/2} + C_1$$

$$= \frac{2}{3}(x^2 + 3x - 18)^{3/2} + C_1$$

and $I_2 = \int \sqrt{x^2 + 3x - 18}\, dx$

$$= \int \sqrt{x^2 + 3x + \frac{9}{4} - 18 - \frac{9}{4}}\, dx$$

$$= \int \sqrt{\left(x + \frac{3}{2}\right)^2 - \left(\frac{9}{2}\right)^2}\, dx$$

$$= \frac{1}{2}\left(x + \frac{3}{2}\right)\sqrt{\left(x+\frac{3}{2}\right)^2 - \left(\frac{9}{2}\right)^2}$$

$$- \frac{1}{2}.\left(\frac{9}{2}\right)^2 \log\left\{x + \frac{3}{2} + \sqrt{\left(x + \frac{3}{2}\right)^2 - \left(\frac{9}{2}\right)^2}\right\} + C_2$$

$$= \frac{2x+3}{4}\sqrt{x^2 + 3x - 18}$$

$$- \frac{81}{8}\log\left\{\frac{2x+3}{2} + \sqrt{x^2 + 3x - 18}\right\} + C_2$$

$\therefore \quad I = \frac{1}{2}\left[\frac{2}{3}(x^2+3x-18)^{3/2}+4\right] - \frac{9}{2}\left[\frac{2x+3}{4}\right.$

$\left.\sqrt{x^2+3x-18} - \frac{81}{8}\log\left\{\frac{2x+3}{2}+\sqrt{x^2+3x-18}\right\}\right] + C_2$

$\Rightarrow \quad I = \frac{1}{3}(x^2+3x-18)^{3/2} - \frac{9}{8}(2x+3)$

$\sqrt{x^2+3x-18} + \frac{729}{16}\log\left\{\frac{2x+3}{2}+\sqrt{x^2+3x-18}\right\}$
$+ C,$

where $C = \frac{C_1}{2} - \frac{9C_2}{2}$

Q. 2. Evaluate : $\int \frac{\sqrt{1-\sin x}}{1+\cos x} e^{-x/2}\, dx.$

[*CBSE*, 2013 (*Comptt.*)]

Solution : $I = \int \frac{\sqrt{1-\sin x}}{1+\cos x} e^{-x/2}\, dx$

$= \int \frac{\sqrt{\cos^2\frac{x}{2}+\sin^2\frac{x}{2}-2\sin\frac{x}{2}\cos\frac{x}{2}}}{\cos^2\frac{x}{2}+\sin^2\frac{x}{2}+\cos^2\frac{x}{2}-\sin^2\frac{x}{2}} e^{-x/2}\, dx$

$= \int \frac{\cos\frac{x}{2}-\sin\frac{x}{2}}{2\cos^2\frac{x}{2}} e^{-x/2}\, dx$

$= \frac{1}{2}\int \frac{e^{-x/2}}{\cos x/2}\, dx - \frac{1}{2}\int \underset{\text{II}}{\sec x/2 \tan x/2}\ \underset{\text{I}}{e^{-x/2}dx}$

$= \frac{1}{2}\int \sec x/2\ e^{-x/2}\, dx - \frac{1}{2}\left[e^{-x/2}\left(2\sec\frac{x}{2}\right)\right.$

$\left.-\int e^{-x/2}\left(-\frac{1}{2}\right)\left(2\sec\frac{x}{2}\right)dx\right]$

$= -e^{-x/2}\sec\frac{x}{2} + C$

Q. 3. Evaluate : $\int \frac{5x-2}{1+2x+3x^2}\, dx.$

[*CBSE*, 2014 (*Comptt.*)]

Solution : $I = \int \frac{5x-2}{1+2x+3x^2}\, dx$

$= \int \frac{\frac{5}{6}(6x+2)-2-\frac{5}{3}}{3x^2+2x+1}\, dx$

$= \frac{5}{6}\int \frac{6x+2}{3x^2+2x+1}\, dx - \frac{11}{3}\int \frac{1}{3x^2+2x+1}\, dx$

$= \frac{5}{6}\log(3x^2+2x+1) - \frac{11}{9}\int \frac{1}{x^2+\frac{2}{3}x+\frac{1}{3}}\, dx$

$= \frac{5}{6}\log(3x^2+2x+1) - \frac{11}{9}\int \frac{1}{\left(x+\frac{1}{3}\right)^2+\frac{1}{3}-\frac{1}{9}}\, dx$

$= \frac{5}{6}\log(3x^2+2x+1) - \frac{11}{9}\int \frac{1}{\left(x+\frac{1}{\sqrt{3}}\right)^2+\left(\frac{\sqrt{2}}{3}\right)^2}$

$= \frac{5}{6}\log(3x^2+2x+1) - \frac{11}{9}\cdot\frac{1}{\frac{\sqrt{2}}{3}}\tan^{-1}\left(\frac{x+\frac{1}{3}}{\frac{\sqrt{2}}{3}}\right) + C$

$= \frac{5}{6}\log(3x^2+2x+1) - \frac{11}{3\sqrt{2}}\tan^{-1}\left(\frac{3x+1}{\sqrt{2}}\right) + C$

Q. 4. Evaluate : $\int\left(\sqrt{\cot x}+\sqrt{\tan x}\right)dx.$

[*AI CBSE*, 2013; *CBSE*, 2014 (*Comptt.*)]

Solution : $I = \int\left(\sqrt{\cot x}+\sqrt{\tan x}\right)dx$

$= \int \sqrt{\tan x}\,(1+\cot x)\, dx$

$= \int \sqrt{\tan x}\left(1+\frac{1}{\tan x}\right)dx$

(Put $\tan x = t^2 \Rightarrow \sec^2 x\, dx = 2t\, dt$;

$\therefore dx = \frac{2t\,dt}{\sec^2 x} = \frac{2t\,dt}{1+\tan^2 x} = \frac{2t\,dt}{1+t^4}$)

$\therefore \quad I = \int \sqrt{t^2}\left(1+\frac{1}{t^2}\right)\frac{2t\,dt}{1+t^4}$

$= 2\int \frac{t^2+1}{t^4+1}\, dt$

$= 2\int \frac{1+\frac{1}{t^2}}{t^2+\frac{1}{t^2}}\, dt$

$= \int \frac{1+\frac{1}{t^2}}{\left(t-\frac{1}{t}\right)^2+2}\, dt$

(Put $t - \frac{1}{t} = z \Rightarrow \left(1+\frac{1}{t^2}\right)dt = dz$)

$= 2\int \frac{dz}{z^2+2}$

$= 2\int \frac{1}{z^2+(\sqrt{2})^2}\, dz$

$= 2\cdot\frac{1}{\sqrt{2}}\tan^{-1}\left(\frac{z}{\sqrt{2}}\right) + C$

$= \sqrt{2}\tan^{-1}\left(\frac{t-\frac{1}{t}}{\sqrt{2}}\right) + C$

$= \sqrt{2}\tan^{-1}\left(\frac{t^2-1}{\sqrt{2}\,t}\right) + C$

$= \sqrt{2}\tan^{-1}\left(\frac{\tan x-1}{\sqrt{2\tan x}}\right) + C$

Q. 5. Evaluate : $\int \frac{1}{\sin^4 x+\sin^2 x\cos^2 x+\cos^4 x}\, dx.$

(*AI CBSE*, 2014)

Solution : $I = \int \frac{1}{\sin^4 x + \sin^2 x \cos^2 x + \cos^4 x}$

$$= \int \frac{\sec^4 x}{1 + \tan^2 x + \tan^4 x} dx$$

$$= \int \frac{\sec^2 x \sec^2 x}{1 + \tan^2 x + \tan^4 x} dx$$

$$= \int \frac{(1 + \tan^2 x) \sec^2 x}{1 + \tan^2 x + \tan^4 x} dx$$

(Put $\tan x = t \Rightarrow \sec^2 x\, dx = dt$)

$$= \int \frac{1 + t^2}{1 + t^2 + t^4} dt$$

$$= \int \frac{1 + \frac{1}{t^2}}{t^2 + \frac{1}{t^2} + 1} dt$$

$$= \int \frac{1 + \frac{1}{t^2}}{\left(t - \frac{1}{t}\right)^2 + (\sqrt{3})^2} dt$$

(Put $t - \frac{1}{t} = z \Rightarrow \left(1 + \frac{1}{t^2}\right) dt = dz$),

$$= \int \frac{dz}{z^2 + (\sqrt{3})^2}$$

$$= \frac{1}{\sqrt{3}} \tan^{-1}\left(\frac{z}{\sqrt{3}}\right) + C$$

$$= \frac{1}{\sqrt{3}} \tan^{-1}\left(\frac{t - \frac{1}{t}}{\sqrt{3}}\right) + C$$

$$= \frac{1}{\sqrt{3}} \tan^{-1}\left(\frac{t^2 - 1}{\sqrt{3}t}\right) + C$$

$$= \frac{1}{\sqrt{3}} \tan^{-1}\left(\frac{\tan^2 x - 1}{\sqrt{3} \tan x}\right) + C$$

Q. 6. Evaluate : $\int \frac{1}{\cos^4 x + \sin^4 x} dx$. *(AI CBSE, 2014)*

Solution : $I = \int \frac{1}{\cos^4 x + \sin^4 x} dx$

$$= \int \frac{\sec^4 x}{1 + \tan^4 x} dx$$

$$= \int \frac{\sec^2 x \sec^2 x}{1 + \tan^4 x} dx$$

$$= \int \frac{(1 + \tan^2 x) \sec^2 x}{1 + \tan^4 x} dx$$

(Put $\tan x = t \Rightarrow \sec^2 x\, dx = dt$)

$$= \int \frac{1 + t^2}{1 + t^4} dt$$

$$= \int \frac{1 + \frac{1}{t^2}}{t^2 + \frac{1}{t^2}} dt$$

$$= \int \frac{1 + \frac{1}{t^2}}{\left(t - \frac{1}{t}\right)^2 + 2} dt$$

[Put $t - \frac{1}{t} = z \Rightarrow \left(1 + \frac{1}{t^2}\right) dt = dz$]

$$= \int \frac{dz}{z^2 + 2}$$

$$= \int \frac{dz}{z^2 + (\sqrt{2})^2}$$

$$= \frac{1}{\sqrt{2}} \tan^{-1} \frac{z}{\sqrt{2}} + C$$

$$= \frac{1}{\sqrt{2}} \tan^{-1} \frac{t - \frac{1}{t}}{\sqrt{2}} + C$$

$$= \frac{1}{\sqrt{2}} \tan^{-1}\left(\frac{t^2 - 1}{\sqrt{2}\, t}\right) + C$$

$$= \frac{1}{\sqrt{2}} \tan^{-1}\left(\frac{\tan^2 x - 1}{\sqrt{2} \tan x}\right) + C$$

Q. 7. Evaluate : $\int \frac{x + 2}{\sqrt{x^2 + 5x + 6}} dx$. *(AI CBSE, 2014)*

Solution : $I = \int \frac{x + 2}{\sqrt{x^2 + 5x + 6}} dx$

Let $x + 2 = A \frac{d}{dx}(x^2 + 5x + 6) + B$

$\Rightarrow$ $x + 2 = A(2x + 5) + B$

$\Rightarrow$ $x + 2 = 2Ax + (5A + B)$

Comparing the coefficients, we get

$2A = 1$

$5A + B = 2$

$\Rightarrow$ $A = \frac{1}{2}, B = -\frac{1}{2}$

$$\therefore I = \frac{1}{2} \int \frac{2x + 5}{\sqrt{x^2 + 5x + 6}} dx - \frac{1}{2} \int \frac{dx}{\sqrt{x^2 + 5x + 6}}$$

$$= I_1 - I_2$$

where $I_1 = \frac{1}{2} \int \frac{2x + 5}{\sqrt{x^2 + 5x + 6}} dx$

[Put $x^2 + 5x + 6 = t \Rightarrow (2x + 5)\, dx = dt$]

$$= \frac{1}{2} \int \frac{dt}{\sqrt{t}}$$

$$= \sqrt{t} + C_1$$

$$= \sqrt{x^2 + 5x + 6} + C_1$$

and $$I_2 = \frac{1}{2}\int \frac{dx}{\sqrt{x^2+5x+6}}$$

$$= \frac{1}{2}\int \frac{dx}{\sqrt{x^2+5x+\frac{25}{4}+6-\frac{25}{4}}}$$

$$= \frac{1}{2}\int \frac{dx}{\sqrt{\left(x+\frac{5}{2}\right)^2-\left(\frac{1}{2}\right)^2}}$$

$$= \frac{1}{2}\log\left\{x+\frac{5}{2}+\sqrt{\left(x+\frac{5}{2}\right)^2-\left(\frac{1}{2}\right)^2}\right\}+C_2$$

$$= \frac{1}{2}\log\left\{x+\frac{5}{2}+\sqrt{x^2+5x+6}\right\}+C_2$$

$\therefore$ $$I = \sqrt{x^2+5x+6}+C_1-\frac{1}{2}\log\left\{x+\frac{5}{2}+\sqrt{x^2+5x+6}\right\}-C_2$$

$$= \sqrt{x^2+5x+6}-\frac{1}{2}\log\left\{x+\frac{5}{2}+\sqrt{x^2+5x+6}\right\}+C$$

where $C = C_1 - C_2$

Q. 8. Evaluate : $\int \frac{x+2}{\sqrt{x^2+2x+3}}\,dx$. *(AI CBSE, 2013)*

Solution : $$I = \int \frac{x+2}{\sqrt{x^2+2x+3}}\,dx$$

$$= \int \frac{x+1+1}{\sqrt{x^2+2x+3}}\,dx$$

$$= \int \frac{x+1}{\sqrt{x^2+2x+3}}\,dx + \int \frac{1}{\sqrt{x^2+2x+3}}\,dx$$

$= I_1 + I_2$ (say)

where $$I_1 = \int \frac{x+1}{\sqrt{x^2+2x+3}}\,dx$$

(Put $x^2+2x+3 = t \Rightarrow (2x+2)\,dx = dt$

$\Rightarrow 2(x+1)\,dx = dt \Rightarrow (x+1)\,dx = \frac{1}{2}\,dt$)

$$= \frac{1}{2}\int \frac{dt}{\sqrt{t}}$$

$$= \frac{1}{2}\cdot\frac{t^{1/2}}{1/2}+C_1$$

$$= \sqrt{t}+C_1$$

$$= \sqrt{x^2+2x+3}+C_1$$

and $$I_2 = \int \frac{1}{\sqrt{x^2+2x+3}}\,dx$$

$$= \int \frac{1}{\sqrt{(x+1)^2+(\sqrt{2})^2}}$$

$$= \log\left\{(x+1)+\sqrt{(x+1)^2+(\sqrt{2})^2}\right\}+C_2$$

$$= \log\left\{(x+1)+\sqrt{x^2+2x+3}\right\}+C_2$$

$$\therefore I = \sqrt{x^2+2x+3}+C_1+\log\left\{(x+1)+\sqrt{x^2+2x+3}\right\}+C_2$$

$$= \sqrt{x^2+2x+3}+\log\left\{(x+1)+\sqrt{x^2+2x+3}\right\}+C$$

where $C = C_1 + C_2$

Q. 9. Evaluate : $\int \frac{x\sin^{-1}x}{\sqrt{1-x^2}}\,dx$.

(CBSE, Outside Delhi, 2012; USEB, 2014)

Solution : Let $$I = \int \frac{x\sin^{-1}x}{\sqrt{1-x^2}}$$

Putting $\sin^{-1}x = t$

then $\frac{1}{\sqrt{1-x^2}}\,dx = dt$

and also $x = \sin t$

$\therefore$ $I = \int t.\sin t\,dt$

Now integrating by parts, we get

$$I = t(-\cos t) - \int 1.(-\cos t)\,dt + C$$

$$= -t\cos t + \int \cos t\,dt + C$$

$$= -t\cos t + \sin t + C$$

$$= -\sin^{-1}x.\sqrt{1-x^2}+x+C$$

Q. 10. Evaluate : $\int \frac{5x-2}{1+2x+3x^2}\,dx$. *(CBSE, 2013)*

Solution : $$I = \int \frac{5x-2}{1+2x+3x^2}\,dx$$

Let $5x-2 = A\frac{d}{dx}(1+2x+3x^2)+B$

$\Rightarrow$ $5x-2 = A(2+6x)+B$

Comparing the coefficients, we get

$6A = 5$

$2A + B = -2$

Solving these, we get

$$A = \frac{5}{6},\ B = -\frac{11}{3}$$

$\therefore$ $$I = \int \frac{\frac{5}{6}(2+6x)-\frac{11}{3}}{1+2x+3x^2}\,dx$$

$$= \frac{5}{6}\int \frac{2+6x}{1+2x+3x^2}\,dx - \frac{11}{3}\int \frac{1}{1+2x+3x^2}\,dx$$

$$= \frac{5}{6}\log(1+2x+3x^2) - \frac{11}{9}\int \frac{1}{x^2+\frac{2x}{3}+\frac{1}{3}}\,dx$$

$$= \frac{5}{6}\log(1+2x+3x^2) - \frac{11}{9}\int \frac{1}{x^2+\frac{2}{3}x+\frac{1}{9}+\frac{1}{3}-\frac{1}{9}}\,dx$$

$$= \frac{5}{6} \log(1 + 2x + 3x^2) - \frac{11}{9} \int \frac{1}{\left(x + \frac{1}{3}\right)^2 + \left(\frac{\sqrt{2}}{3}\right)^2} dx$$

$$= \frac{5}{6} \log(1 + 2x + 3x^2) - \frac{11}{9} \frac{1}{\frac{\sqrt{2}}{3}} \tan^{-1} \left(\frac{x + \frac{1}{3}}{\frac{\sqrt{2}}{3}}\right) + C$$

$$= \frac{5}{6} \log(1 + 2x + 3x^2) - \frac{11}{9} \times \frac{1}{\frac{\sqrt{2}}{3}} \tan^{-1} \left(\frac{3x + 1}{\sqrt{2}}\right) + C$$

Q. 11. Evaluate : $\int \frac{x + 3}{\sqrt{5 - 4x + x^2}} dx.$ (*USEB*, 2014)

Solution : $I = \int \frac{x + 3}{\sqrt{5 - 4x + x^2}} dx$

$$= \int \frac{x + 3}{\sqrt{5 + x^2 - 4x + 4 - 4}} dx$$

$$= \int \frac{x + 3}{\sqrt{1 + (x - 2)^2}} dx$$

$$= \int \frac{x - 2 + 5}{\sqrt{1 + (x - 2)^2}} dx$$

$$= \int \frac{x - 2}{\sqrt{1 + (x - 2)^2}} dx + 5 \int \frac{dx}{\sqrt{1 + (x - 2)^2}}$$

$= I_1 + 5I_2$ where,

$$I_1 = \int \frac{x - 2}{\sqrt{1 + (x - 2)^2}} dx$$

(Put $1 + (x - 2)^2 = t^2 \Rightarrow 2(x - 2)\, dx = 2t\, dt$
$\Rightarrow (x - 2)\, dx = t\, dt$)

$$= \int \frac{t\, dt}{t} = \int dt$$

$$= t + C_1$$

$$= \sqrt{1 + (x - 2)^2} + C_1$$

$$I_2 = \log\left\{x - 2 + \sqrt{1 + (x - 2)^2}\right\} + C_2$$

$\therefore$ $I = \sqrt{1 + (x - 2)^2} + C_1 + 5 \log \left\{x - 2 + \sqrt{5 - 4x + x^2}\right\} + 5C_2$

$$= \sqrt{5 - 4x + x^2} + 5 \log \left\{x - 2 + \sqrt{5 - 4x + x^2}\right\} + C$$

where $C = C_1 + 5C_2$

Q.12. Evaluate : $\int \frac{\sqrt{x^2 + 1}\,[\log(x^2 + 1) - 2\log x]}{x^4} dx.$

[*BSER*, 2013; *AI CBSE*, 14 (*Comptt.*)]

Solution :

$$I' = \int \frac{\sqrt{x^2 + 1}\left[\log(x^2 + 1) - 2\log x\right]}{x^4} dx$$

$$= \int \sqrt{\frac{x^2 + 1}{x^2}} \log \frac{x^2 + 1}{x^2} \cdot \frac{1}{x^3} dx$$

$$= \int \sqrt{1 + \frac{1}{x^2}} \log\left(1 + \frac{1}{x^2}\right) \cdot \frac{1}{x^3} dx$$

(Put $1 + \frac{1}{x^2} = t^2 \Rightarrow \frac{2}{x^3} dx = 2t\, dt$
$\Rightarrow \frac{1}{x^3} dx = -t\, dt$)

$$\therefore I = \int \sqrt{t^2} \log t^2 (-t)\, dt$$

$$= -\int t^2 \log t^2\, dt$$

$$= -\int t^2\, 2 \log t\, dt$$

$$= -2 \int \underset{\text{II}}{t^2} \underset{\text{I}}{\log t}\, dt$$

$$= -2\left[\log t \frac{t^3}{3} - \int \frac{1}{t} \cdot \frac{t^3}{3} dt\right]$$

$$= -\frac{2}{3} t^3 \log t + \frac{2}{3} \int t^2\, dt$$

$$= -\frac{2}{3} t^3 \log t + \frac{2}{3} \frac{t^3}{3} + C$$

$$= -\frac{2}{3} t^3 \log t + \frac{2}{9} t^3 + C$$

$$= -\frac{2}{3}\left(1 + \frac{1}{x^2}\right)^{3/2} \log \sqrt{1 + \frac{1}{x^2}} + \frac{2}{9}\left(1 + \frac{1}{x^2}\right)^{3/2} + C$$

$$= \frac{2}{3}\left(1 + \frac{1}{x^2}\right)^{3/2} \left[\frac{1}{3} - \log \sqrt{1 + \frac{1}{x^2}}\right] + C$$

Q. 13. Integrate : $\int e^x \cos x\, dx.$ (*BSEB*, 2015)

Solution :

Let $I = \int e^x \cos x\, dx$

$\Rightarrow$ $I = e^x \int \cos x\, dx - \int\left\{\frac{d}{dx}(e^x) \cdot \int \cos x\, dx\right\} dx$

$\Rightarrow$ $I = e^x(-\sin x) - \int e^x(-\sin x)\, dx$

$\Rightarrow$ $I = -e^x \sin x + e^x \int \sin x\, dx - \int\left\{\frac{d}{dx}(e^x) \cdot \int \sin x\, dx\right\} dx$

$\Rightarrow$ $I = -e^x \sin x + e^x \cos x - \int e^x \cos x\, dx$

$\Rightarrow$ $I = e^x \cos x - e^x \sin x - I$

$\Rightarrow$ $2I = e^x(\cos x - \sin x)$

$\Rightarrow$ $I = \frac{e^x}{2}(\cos x - \sin x)$

$$\therefore \int e^x \cos x\, dx = \frac{e^x}{2}(\cos x - \sin x)$$

Q. 14. Evaluate : $\int \sqrt{1+\cos 2x}\, dx$ (*Raj. Board*, 2015)

Solution : $\int \sqrt{1+\cos 2x}\, dx$

$$= \int \sqrt{2\cos^2 x}\, dx$$
$$= \sqrt{2}\int \cos x\, dx$$
$$= \sqrt{2}\ \sin x + C$$

Q. 15. Evaluate : $\int \frac{dx}{\sqrt{9+8x-x^2}}$ (*Raj. Board*, 2015)

Solution : $\int \frac{dx}{\sqrt{9+8x-x^2}}$

$$= \int \frac{dx}{\sqrt{-(x^2-8x-9)}} = \int \frac{dx}{\sqrt{-(x^2-8x+16-25)}}$$
$$= \frac{dx}{\sqrt{-\{(x-4)^2-5^2\}}} = \int \frac{dx}{\sqrt{5^2-(x-4)^2}}$$
$$= \sin^{-1}\left(\frac{x-4}{5}\right) + C$$

Q. 16. Evaluate : $\int x\tan^{-1} x\, dx$ (*Raj. Board*, 2015)

Solution : $\int x\tan^{-1} x\, dx$

$$= (\tan^{-1} x)\ \frac{x^2}{2} - \int \frac{1}{1+x^2}\cdot\frac{x^2}{2}\, dx$$
$$= \frac{x^2}{2}\ \tan^{-1} x - \frac{1}{2}\int \frac{x^2+1-1}{x^2+1}\, dx$$
$$= \frac{x^2}{2}\ \tan^{-1} x - \int\left(1-\frac{1}{x^2+1}\right) dx$$
$$= \frac{x^2}{2}\ \tan^{-1} x - \frac{1}{2}\ (x - \tan^{-1} x) + C$$

Q. 17. Evaluate : $\int e^x \frac{(x^2+1)}{(x+1)^2}\, dx$ (*JAC*, 2015)

Solution : $\int e^x \frac{(x^2+1)}{(x+1)^2}\, dx = \int e^x\left(1-\frac{2x}{(x+1)^2}\right) dx$

$$= \int e^x dx - 2\int e^x \cdot \frac{x}{(x+1)^2}\, dx$$
$$= e^x - 2\ \int e^x \cdot \frac{x+1-1}{(x+1)^2}\, dx$$
$$= e^x - 2\ \int e^x \left\{\frac{1}{x+1} + \frac{-1}{(x+1)^2}\right\} dx$$
$$= e^x - 2\ \left\{\int e^x \cdot \frac{1}{x+1}\, dx - \int e^x\ \frac{1}{(x+1)^2}\, dx\right\}$$
$$= e^x - 2\ \left[\frac{1}{x+1}\cdot e^x - \int -\frac{1}{(x+1)^2}\, e^x dx - \int e^x\ \frac{1}{(x+1)^2}\, dx\right]$$
$$= e^x - 2\left[\frac{1}{x+1}\cdot e^x + \int e^x \cdot \frac{1}{(x+1)^2}\, dx - \int e^x \cdot \frac{1}{(x+1)^2}\, dx\right] + C$$
$$= e^x - 2\frac{e^x}{x+1} + C$$

Q. 18. Evaluate : $\int \frac{e^{\tan^{-1} x}}{1+x^2}\, dx$ (*JAC*, 2015)

Solution : $\int \frac{e^{\tan^{-1} x}}{1+x^2}\, dx$

Let $\tan^{-1} x = t \Rightarrow \frac{1}{1+x^2}\ dx = dt$

$$= \int e^t dx$$
$$= e^t + C$$
$$= e^{\tan-1} x + C$$

Q. 19. Integrate the function :

$$\frac{2x}{1+x^2}$$ (*USEB*, 2015)

Solution : $\int \frac{2x}{1+x^2}\, dx$

Let $1 + x^2 = t \Rightarrow 2x\ dx = dt$

$$= \int \frac{1}{t}\, dx$$
$$= \log t + C$$
$$= \log (1 + x)^2 + C$$

NCERT QUESTIONS

Q. 1. Evaluate : $\int \frac{\sec^2 x}{\operatorname{cosec}^2 x}\, dx.$

(*CBSE Delhi*, 2011; *BSER*, 2014)

Solution :

$$\int \frac{\sec^2 x}{\operatorname{cosec}^2 x}\ dx = \int \frac{\sin^2 x}{\cos^2 x}\ dx$$
$$= \int \tan^2 x\, dx$$
$$= \int (\sec^2 x - 1)\ dx$$
$$= \tan x - x + C$$

Q. 2. Evaluate : $\int \frac{1}{\cos (x-a).\cos (x-b)}\, dx.$

Solution : $\frac{1}{\sin (b-a)} \int \frac{\sin (x-a) - (x-b)}{\cos (x-a) \cos (x-b)}\, dx$

$$= \frac{1}{\sin (b-a)} \int \frac{\sin (x-a) \cos\ (x-b)}{\cos (x-a) \cos (x-b)} - \frac{\cos (x-a) \sin (x-b)}{\cos (x-a) \cos (x-b)}\ dx$$
$$= \frac{1}{\sin (b-a)} \int \{\tan (x-a) - \tan (x-b)\}\ dx$$
$$= \frac{1}{\sin (b-a)} \int [\log \{\sec (x-a)\} - \log \{\sec x\,(x-b)\}] + C$$
$$= \frac{1}{\sin (b-a)}\ \log\left[\frac{\sec (x-a)}{\sec (x-b)}\right] + C$$
$$= \frac{1}{\sin (a-b)}\ \log\left[\frac{\cos (x-a)}{\cos (x-b)}\right] + C$$